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ROSSITER'S
TRIGONOMETRY.

THE ELEMENTS OF PLANE AND SPHERICAL TRIGONOMETRY.

THE

ELEMENTS

OF

PLANE AND SPHERICAL TRIGONOMETRY.

BY

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BODLEIANA · BIBLIOTHECA ·

LONDON: WILLIAM TEGG.
1868.

183. g. 29.

PRINTED BY W. CLOWES AND SONS, STAMFORD STREET
AND CHARING CROSS.

PREFACE.

The purpose of the present treatise is to give a clear idea of Trigonometry, both Plane and Spherical, as a science, and also to explain its practical utility. The whole of the book is entirely original, and the examples have been written with a view to illustrate the principles of the science and their practical application, rather than to puzzle the student by an array of difficult formulæ.

The series of tables at the end will, it is hoped, be found very useful.

W. Rossiter.

Linden Villa,
Tottenham, N.

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ABBREVIATIONS AND DEFINITIONS.

Sine	is usually	written	*sin.*
Cosine	,,	,,	*cos.*
Tangent	,,	,,	*tan.*
Cotangent	,,	,,	*cot.*
Secant	,,	,,	*sec.*
Cosecant	,,	,,	*cosec.*

In any triangle the angles are usually expressed by A, B, and C, and the sides respectively opposite to these by *a*, *b*, and *c*.

Sin × *sin* is usually written *sin*^{2}, and so of all the other ratios.

Angle $(A + B)$ means an angle equal to the sum of the two angles A and B. If $A = 40^\circ$ and $B = 35^\circ$, then $A + B = 75^\circ$.

Angle $(A - C)$ means the angle left when C is taken from A. If $A = 40^\circ$ and $C = 35^\circ$, then $A - C = 5^\circ$.

By "Solution of a Triangle," is meant the finding the values of its three sides and three angles. To do this it is necessary to have given the three sides, two sides and an angle, or one side and two angles.

INTRODUCTION.

The study of Trigonometry requires a knowledge of Algebra and Geometry. Theoretically, it is the study of the relations existing between the different parts of triangles generally; practically, it is the application of the propositions so established to the purpose of calculating, from given data, the magnitude of the sides and angles of particular triangles.

To commence this study with any hope of proficiency, it is requisite to be familiar with Geometry, especially with the Propositions established in the Sixth Book of Euclid, and also to have a practical knowledge of Algebra, at least as far as the solution of Quadratic Equations.

In the edition of Euclid,[*] which is intended as an introduction to Trigonometry, the Second Book is discussed Algebraically, as being more completely understood, and also more practically available, when so treated. In the same manner it is proposed here to discuss the Sixth Book of Euclid's Elements, which is the one upon which Trigonometry is more especially established.

Prop. 1. Triangles and parallelograms of the same altitude are to one another as their bases.

Let there be any number of triangles of equal altitude, then these triangles are to each other as their bases.

Let m be the common altitude of any two of the triangles A B C and D E F, and the respective bases be a and b. Then, if $a = x\,b$, also the triangle A B C $= x$ times the triangle D E F.

---

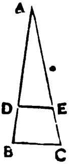
[*] Euclid's Elements of Geometry. Tegg and Co., London, 1867.

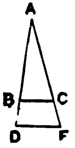
For a may be divided in x parts, each one equal b. Let each of these points of division be joined with the vertical angle, then there will be x triangles, each having a base equal to the base of b, and being between the same parallels.

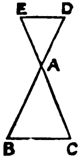
$\therefore$ triangle A B C $= x$ triangles D E F;
i.e., triangle A B C : triangle D E F : : $a : b$.

Prop. 2 is a very important one.

If a line be drawn parallel to any side of a triangle, it will cut the other sides, produced if necessary, in the same ratio.







Let a, b, c, be the sides of the triangle, opposite respectively to A, B, and C; then, if D E be drawn parallel to a, it will divide b and c, in the same ratio.

Let A D be x, and A E be y, then D B $= c - x$, and E C $= b - y$, and the theorem to be established is that—

$$b : c :: b - y : c - x;$$
also, $$c : b :: c - x : b - y,$$
and $$b : c :: y : x.$$

If, therefore, a line be drawn parallel to a, bisecting c it also bisects b; if it cuts off one-third of c it also cuts off one-third of b, &c.

It will be seen that the line need not be drawn within the triangle; in (2) it is drawn below the base, and in (3) above the vertex; in each case the sides are produced to meet it, and the theorem still holds good. But in (2) x and y are greater, respectively, than c and b, and the theorem becomes—

$$x : y :: c : b;$$

also, $$x - c : y - b :: c : b,$$

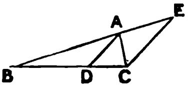
and $$y : x :: b : c.$$

In (3), x and y are distinct from c and b, and may be of equal, greater, or less magnitude than c and b respectively; then the theorem becomes—

$$x : y :: c : b,$$

and $$c + x : b + y :: c : b.$$

Prop. 3 is that, if an angle of any triangle be bisected, the bisecting line cuts the opposite side of the triangle into two parts, having the same ratio as the other two sides.



Express the sides by a, b, and c, respectively, and—

Let the angle A be bisected by the line A D.

Then let C D = x, and B D = $a - x$, then—

$$c : b :: a - x : x.$$

Produce B A to E to meet C E drawn parallel to A D, then—

$$\text{A E} = \text{A C} = b.$$

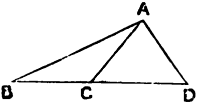
For angle A C E = angle A E C.

Then $$\text{B A} : \text{A E} :: \text{B D} : \text{D C};$$

i.e., $$c : b :: a - x : x.$$

Prop. A is a peculiar case of the general theorem of Prop. 3, in which the angle bisected is an external one, and the base is produced to meet the cutting line.

In this case—



x is greater than a,

and the proposition becomes—

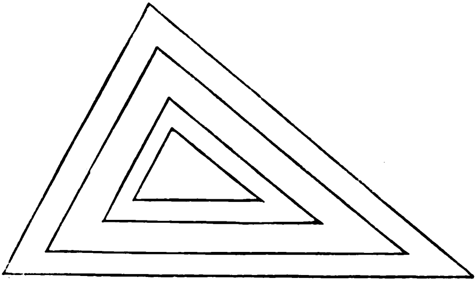
$$x : a + x : : c : b.$$

If x is considered $-x$ when it is on the left of C, and $+x$ when to the right of it, the general formula

$$c : b : : x : a + x$$

will apply to both cases.

Prop. 4 is a very important one, and is of very great use in establishing some of the fundamental propositions of Trigonometry.



If there be any number of equiangular triangles, the sides about the equal angles are proportionals.

Let A, B, and C, be the three angles; and the sides of the different triangles be, respectively, a, b, c; a', b', c'; a'', b'', c'', &c.; then the proposition is, that—

$$a : b :: a' : b' :: a'' : b'', \text{ \&c.},$$
$$a : c :: a' : c' :: a'' : c'', \text{ \&c.},$$
and
$$b : c :: b' : c' :: b'' : c''.$$

This may also be expressed, as—

$$\frac{a}{b} = \frac{a'}{b'} = \frac{a''}{b''},$$
$$\frac{a}{c} = \frac{a'}{c'} = \frac{a''}{c''},$$
and
$$\frac{b}{c} = \frac{b'}{c'} = \frac{b''}{c''}.$$

Prop. 5 is the converse of this, and establishes, that if—

$$\frac{a}{b} = \frac{a'}{b'}, \text{ and } \frac{a}{c} = \frac{a'}{c'};$$
then
$$A = A', B = B', \text{ and } C = C'.$$

Props. 6 and 7 establish the same theorem of equality between the relative parts of two triangles upon a still narrower basis. Let A B C $a\,b\,c$, and A′ B′ C′ $a'\,b'\,c'$ be two triangles;

then, if $$A = A', \text{ and } \frac{b}{c} = \frac{b'}{c'};$$

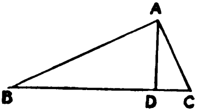
also, $$B = B', C = C', \text{ and } \frac{a}{c} = \frac{a'}{c'}.$$

Again, if $$A = A' \text{ and } \frac{a}{b} = \frac{a'}{b'}, \text{ or } \frac{a}{c} = \frac{a'}{c'};$$

then also, $$B = B', C = C', \text{and } \frac{b}{c} = \frac{b'}{c'};$$

that is, in either case the two triangles are equiangular. The one exception to the truth of this theorem is discussed in Part II., Chapter IV.

PROP. 8 is a theorem that any right-angled triangle is divided into two other right-angled triangles, similar to the first, if a line be drawn from the right angle perpendicular to the base.



Let the sides opposite A, B, C, be, respectively, a, b, c; also let $BD = x$ and $AD = d$. Then—

Triangle A B C is similar to triangle A B D and to A D C,

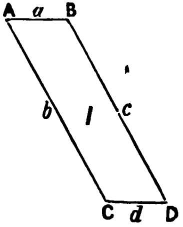
and $BC : AB : AC :: AC : AD : DC,$

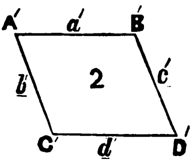
$:: AB : BD : AD;$

i.e., $a : c : b :: b : d : a-x :: c : x : d.$

PROPS. 9 to 13 are Problems, and are not so important for our present purpose as the Theorems. They are solved by means of the theorem that three parallel lines cut all lines crossing them in the same ratio.

PROP. 14 is respecting equal and equiangular parallelograms, showing that their sides are in reciprocal proportion.





Let A B = a, A′ B′ = a', and A C = b, A′ C′ = b'

Then, $a : a' :: b' : b$;

or, $\frac{a}{b'} = \frac{a'}{b}$, and $\frac{a}{a'} = \frac{b'}{b}$.

Prop. 15 proves that what has just been proved for equal parallelograms is also true of equal triangles; thus, if angle A be contained by sides B and C, and angle A′ by sides B′ and C′, then—

$$a : a' :: b' : b.$$

Props. 16 and 17 establish the truth, that when we have four lines in proportion, as above, then—

$$a\,b = a'\,b',$$

and if $a = b$;

then, $a' : a :: b : b'$,

$:: a : b'$,

and $a'\,b' = a^2$.

Prop. 18 is a problem to describe a figure of the same shape as another, and to place it in a given situation with respect to a given line.

Prop. 19 establishes what is the ratio between similar triangles. If a, b, c, and a', b', c', be the two similar triangles, and—

$$a = m\,a';$$

then the area of the one triangle is m^2 times that of the other; thus,—

if $a = 2\,a'$,

the area of one triangle = 4 times the area of the other;

if $a = 5\,a'$,

then one area = 25 times the other, &c.

Prop. 20 extends this theorem to similar polygons.

Prop. 22. If there be four lines, a, b, c, d, so that

$a : b :: c : d$, and there be a triangle, polygon, parallelogram, or any other rectilineal figure described on each of a and b, these figures shall be in the same ratio as any two similar figures, whether triangles, parallelograms, or polygons, described on c and d.

Prop. 23. If there be two equiangular parallelograms, A and A′, contained, respectively, by sides a, b, and a', b'; then—

$$\text{area } A : \text{area } A' :: a\,b : a'\,b'.$$

Props. 24 and 26. If lines be drawn parallel to the sides of any parallelogram, and forming other parallelograms, all such parallelograms are similar to each other and to the first.

Prop. 25. A problem to describe a rectilineal figure, of which the size and shape are both given.

Props. 27, 28, and 29, are not very useful.

Prop. 30. Problem. Let there be a line a, it is required to find a point of division, so that—

$$a : a-x :: a-x : x.$$

Prop. 31. If there be described on the sides a, b, c, of any right-angled triangle, three similar figures, A, B, C, then the figure on the side C opposite the right angle has an area equal to the two areas of the figures on the other sides; *i.e.*,

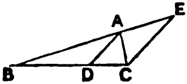
$$C = A + B.$$

This is an important extension of the 47th Prop. of Book I., in which the figures are squares.

Prop. 33 proves that angles in circles are proportional to the arcs on which they stand; that is, if one arc be m times another, the angle on it is also m times the angle on the other.

Props. B, C, and D, are theorems concerning the rectangles contained by lines and portions of lines forming figures within circles, and every triangle may be considered as inscribed within a circle (for since a circle can be drawn through any three given points, one can always be drawn through the three angular points of a triangle).

We saw, in Prop. 3, that in triangle A B C,

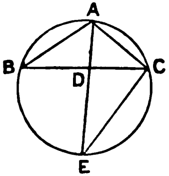


if B D = x, then—

$$c : b :: x : a - x.$$

Prop. B shows that—

$$b\,c = x\,(a-x)+d^2. \qquad (AD = d).$$

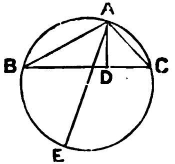


Triangles A B D and A E C are equiangular;

$$\therefore BA : AD :: EA : AC;$$
$$\therefore BA \,.\, AC = AD \,.\, EA,$$
$$= AD\,(AD + DE),$$
$$= AD^2 + AD \,.\, DE,$$
$$= BD \,.\, DC + AD^2;$$
$$\therefore BA \,.\, AC = BD \,.\, DC + AD^2;$$
$$i.e.,\ b\,c = x\,(a-x) + d^2.$$

Prop. C. If the line d, instead of bisecting the angle A, be drawn perpendicular to a, then—

$$b\,c = d \,.\, \textit{diameter of circle};$$



$$\therefore BA : AD :: EA : AC;$$

i.e., $bc = 2dr$. ($r = radius$).

Prop. D. If there be a quadrilateral figure inscribed in a circle, the rectangles of the two pairs of opposite sides equal the rectangle of the diagonals.

If a, b, c, d be in order, the four sides of the quadrilateral, and m, n be the diagonals, then

$$ac + bd = mn.$$

Summing up these three Propositions, B, C, and D, we have the following:—

If there be a line drawn from any angle of a triangle to the opposite side, then,

If the line bisect the angle,

$$bc = x(a - x) + d^2;$$

if the line be perpendicular to the side,

$$bc = d \,.\, \textit{diameter of circle};$$

if there be two triangles on the same chord of a circle, one in either segment, then,

$$ac + bd = mn.$$

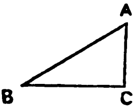
Any of the properties of triangles thus proved may be used as means towards the solution of triangles, as may also any of the theorems of Geometry generally. Familiarity with these theorems will sometimes suggest a ready means of solving problems that appear difficult of solution by the ordinary methods.

TRIGONOMETRY.

CHAPTER I.

In the Treatise on Geometry (Vol. I., p. 12), it was shown that of the six parts of a triangle (three sides and three angles), if any three be given, the remaining three may generally be found. Thus, if we know two sides and an angle of any triangle we may be said to know also the other angles and the remaining side, and *vice versâ.*

Example.—In a right angle the base is 4 inches, and the hypothenuse 5 inches. What is the perpendicular?



Here A B = 5, and B C = 4. Required the length of A C.

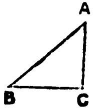
By Euclid, I., 47. $AB^2 = BC^2 + AC^2$;

$$\therefore AC^2 = AB^2 - BC^2 = 25 - 16 = 9;$$

$$\therefore AC = 3.$$

Here, from knowing the length of two sides and the size of one angle, we are able to find the third side. Also from Euclid, I. 32, we know that the two angles A and B are together equal to a right angle; but we are not able, from simple geometry, to find the exact size of each angle, though we know their sum to be 90°.

Example.—A rope reaches from the top of a tree 30 feet high to the ground at 40 feet distance from the tree. How long is the rope?



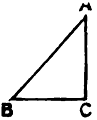
Let A C represent the tree, A B the rope, then B C will be the 40 feet of ground.

By Euclid, I. 47, we know that—

$$A\,B^2 = A\,C^2 + B\,C^2$$
$$= 30^2 + 40^2 = 2500.$$
$$\therefore A\,B = 50, \text{ and the rope is 50 feet long.}$$

Here, as before, we are not able to find the size of either angle A or angle B, though we know that together they amount to 90°; *i. e.*, are equal to a right angle.

Example.—A ladder x feet long makes an angle of 45° when leaning against a wall. What angle does it make with the ground?



Let A B be the ladder, B C the ground, and A C the wall.

By Euclid, I. 32, the three angles, A, B, and C, equal together 180°; and since C = 90°, and A = 45°,

$$\therefore B = 180^\circ = (90^\circ + 45^\circ) = 45^\circ.$$

It follows from this that A and B are equal angles, and, therefore, that A C = B C; but though we know that they are equal, we cannot as yet discover their actual length.

These three examples will illustrate both the general use and the *practical* insufficiency of simple geometry; will show, that while it enables us to establish general theorems respecting space and form, and concerning the *relative* values of lines and angles, it is too indefinite, too comprehensive, to do us much service in the computation of the values of actual lines and angles.

Therefore it is necessary to found, upon these general principles of geometry, some system of practical calculation, by means of which we may solve problems respecting actual triangles, squares, circles, &c., and find not only the *general* and *relative*, but also the *particular* and *specific*, values of lines and angles.

Such a system is Trigonometry, which may be described as a system of practical calculation, founded upon the general theorems of Geometry, by means of which we can apply these general truths to every-day use.

This system is, however (as its name implies), a system applicable only to the calculation of lines and angles forming parts of triangles; but since all figures (including even circles) may be considered as being made up of a number of triangles, this system is applicable much more generally than might at first be supposed.

It must be carefully borne in mind, that there is no essential difference between Geometry and Trigonometry, that no rigid line of distinction can be drawn between them; and it will help us very much in the comprehension of the spirit and value of the operations of Trigonometry if we try to get a clear idea of the relative positions of the three sciences of Geometry, Algebra, and Trigonometry.

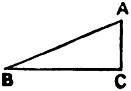
Geometry is the science of form, Algebra the science of number; the one talks of circles, triangles, squares, &c., without the least consideration of their size or extent, the other of numbers, as one, four, twenty, &c., without the least consideration of what the things are; the one science treats only of the shape of things, and the general truths connected with these forms; the other only of their numbers, and the general truths connected with these numbers.

Trigonometry is a fusion of Geometry and Algebra, a science treating not only of general forms, but also of specific numbers; in which we consider a triangle not only as a figure of three sides, but as having sides so many inches, feet, &c., in length: in which we speak not only of one angle being greater than the other, but as being so many degrees and minutes greater.

Thus we have, as the groundwork on which to build our practical science of Trigonometry, the general theorems of Geometry, and these we can apply to our particular figures by the aid of the methods of calculation established by Algebra. Not only this, but we extend our knowledge of the properties of triangles by a more particular and minute study of this class of figures, and eventually we arrive at the power of applying to practical use the general theorem, that in every triangle the angles and sides are so interdependent that if we know any three of the six we may be said to know the others.

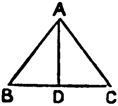
But having done this, we must not imagine that we have exhausted Trigonometry: on the contrary, so far from it being only sufficient for our practical wants, we find it, like all other great truths, extending far beyond not only our needs but our understanding.

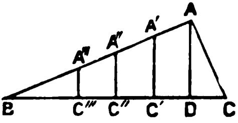
To return to our first problem (page 21).



We found that, A B being 5, and B C being 4, A C was 3; but we were unable to calculate the exact value of the angles A and B, though we knew their sum to be 90°.

This difficulty is solved by an application of the theorem of Euclid, VI. 4, which becomes one of the fundamental theorems of Trigonometry. It might at first be thought that there was some connection between the size of the angle B and the length of the opposite side A C, but a moment's reflection will show this to be not the case. For if so, the angles B and C ought to be equal, since A D belongs as much to one as the other. Also, any number of lines A C, A′ C′, &c., might be drawn from A B to B D, parallel to A D, both nearer to, and farther from, the angle B, and no two of equal length. Therefore it is at once evident that there can be no constant ratio between any angle of a triangle and the opposite side.





Also there can be no constant ratio between the angle and either of the sides containing it, as A B or B C, since the angle B is the same whether the triangle be A B C, A′ B C′, or A‴ B C‴, &c.

But though there is no interdependence between the magnitude of the angle B and the length of A C or of B C, though of any two we may increase or decrease one without there being necessarily any change in the other, yet we shall find that the *three* magnitudes A C, B C, and the angle B are connected together so intimately, and so unalterably, that any change in any one of the three is immediately followed by a change in at least *one* of the others, and by a change in the relation between these others. Thus, we cannot alter A C to the length of A′ C′ without either making the line B C shorter (as B C′) or making the angle B smaller. Also, we cannot shorten B C without also either shortening A C or *increasing* the angle B.

Therefore to *shorten A C* without altering B C is to *decrease the angle at B;* to *shorten B C* without altering A C is to *increase the angle at B;* but if *both* A C and B C be lengthened or shortened *in the same ratio,* the angle B remains unaltered.

Conversely, if we wish to increase the angle B, we may either increase A C, or decrease B C; and if we wish to decrease the angle B, we may either decrease A C or increase B C.

It is quite possible, however, to have the *angle smaller,* though A C and B C be *both* increased; but it will be found in this case that B C is increased much more, in proportion to its own length, than A C; also the *angle* may be *larger,* and A C and B C be both decreased, but B C will be found to be decreased more in proportion to its own length than A C.

We may therefore sum up the connection between the angle B and the sides A C and B C, thus:—

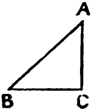
If A C remain the same,
the angle B increases as B C decreases, and
the angle B decreases as B C increases.

If B C remain the same,
 the angle B increases as A C increases, and
 the angle B decreases as A C decreases.

If the angle B remain the same,
 A C and B C increase or decrease together.

From this we see that so long as A C and B C are in the same ratio to each other, the angle B remains the same, and thus we arrive at the notion which is at the foundation of trigonometrical calculations, that the *angles of a triangle depend, not upon the absolute length of the sides of the triangle, but upon their relative lengths, i.e., upon the ratios existing between them.*

Thus, in example 3 (page 22), the ladder makes an angle of 45° with the wall and also with the ground.



Therefore angle A = angle B, and A C = B C (Euclid, I. 6), and from this we may conclude, that whenever the line A C is equal to the line B C, the angle at B will be 45°. Therefore, whenever the perpendicular and the base of a right angled triangle are equal, each of the remaining angles is an angle of 45°; and it is evident that if A C be greater than B C, the angle at B will be more than 45°, and if A C be less than B C, the angle at B will be less than 45°.

We may sum up the relation between the ratio of A C to B C and the angle B, thus:—

$$\text{If } \frac{AC}{BC} = 1, \text{ then angle at B} = 45^\circ.$$

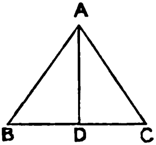
$$\text{If } \frac{A\,C}{B\,C} \reflectbox{$\angle$} 1, \text{ then angle at B } \reflectbox{$\angle$} 45^\circ.$$

$$\text{If } \frac{A\,C}{B\,C} \angle 1, \text{ then angle at B } \angle 45^\circ.$$

But besides the ratio of A C to B C, we find the ratio of A C to A B, *i. e.*, of the perpendicular to the hypothenuse, to be inseparably connected with the magnitude of the angle at B.

If A C = A B, then B C = 0, and the two angles B and C each = 90°, and the triangle becomes a straight line. Therefore, in every right angled triangle the perpendicular must be less than the hypothenuse, *i.e.*—

$$\frac{A\,C}{A\,B} \text{ always less than 1.}$$



Let A B C be a triangle equiangular, and therefore the three angles A, B, and C, each = 60°. Draw A D from A perpendicular to B C, then (Euclid, I. 10) B D = C D, and angle B A D = angle C A D, and therefore B A D and C A D each = 30°.

Now $\quad A B^2 = B D^2 + A D^2 \quad$ (Euclid, I. 47.)
also $\quad A B = 2 B D$;
$\therefore \quad A B : B D : A D :: \sqrt{4} : \sqrt{1} : \sqrt{3} :: 2 : 1 : \sqrt{3}$,
$\therefore$ the ratio of A D to A B is expressed by $\frac{A\,D}{A\,B}$,

which $= \frac{\sqrt{3}}{2}$.

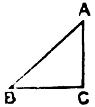
Therefore, when, in a right angled triangle, the ratio of the perpendicular to the hypothenuse is $\frac{\sqrt{3}}{2}$, then the angle opposite the perpendicular is 60°.

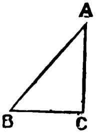
But we may consider B D as the perpendicular, and A D as the base, of the triangle B A D, and the angle opposite the perpendicular will then be B A D = 30°, and

$$\frac{\text{perpendicular}}{\text{hypothenuse}} = \frac{\sqrt{1}}{\sqrt{4}} = \frac{1}{2};$$

$\therefore$ when, in a right angled triangle, the ratio of the perpendicular to the hypothenuse $= \frac{1}{2}$, the angle opposite the perpendicular is 30°.

Collecting the results of these examples, we have:—





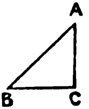
If angle B = 45°, then A C = B C, and $\frac{AC}{BC} = 1$.

If angle B = 60°, then $\frac{AC}{AB} = \frac{\sqrt{3}}{2}$.

If angle B = 30°, then $\frac{AC}{AB} = \frac{1}{2}$.

These three examples will be enough to fix clearly in our minds the fact that, in a triangle, it is the *ratio of the sides to one another*, and *not the length of the sides*, that *determines* the *magnitude of the angles*.

Example.—The perpendicular and base of a right angled triangle are each 6 feet. Required the third side and remaining angles.

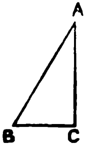


$$A\,C = B\,C = 6. \quad \text{Also } A\,B^2 = A\,C^2 + B\,C^2$$
$$= 36 + 36 = 72;$$
$$\therefore\ A\,B = \sqrt{72} = 8 \cdot 4 +.$$

$$\text{Also } \because\ A\,C = B\,C\ \therefore\ \frac{A\,C}{B\,C} = 1\ \therefore\ \text{angle } B = 45^\circ;$$

$$\text{and } \because\ B\,C = A\,C\ \therefore\ \frac{B\,C}{A\,C} = 1\ \therefore\ \text{angle } A = 45^\circ.$$

Example.—The hypothenuse of a right angled triangle is 8 feet, and the base is 4 feet. Required the perpendicular, and the remaining angles.



$$\frac{B\,C}{A\,B} = \frac{4}{8} = \frac{1}{2}\ \therefore\ \text{angle } A = 30^\circ; \text{ (page 29.)}$$
$$\therefore\ \text{angle } B = 180^\circ - (90^\circ + 30^\circ) = 60^\circ;$$
$$\text{Also } A\,B^2 = A\,C^2 + B\,C^2\ \therefore\ 64 = A\,C^2 + 16\ \cdot$$
$$\therefore\ A\,C^2 = 64 - 16 = 48,$$
$$\text{and} \qquad A\,C = \sqrt{48} = 6 \cdot 9 +.$$

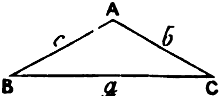
In these two easy examples, we see how, from knowing three out of the six parts of a triangle, we can find the remaining three. These examples are very simple, but they will serve to give us a clear idea of what is meant by the connection between the *angles* and *ratios* of a triangle. Thus the angle B, in the first example, was proved to be 45°; not because A C and B C were each 6 feet, but because they were equal. The angle B would still be 45°, whatever the lengths of A C and B C, so long as these lengths were equal. So also, in the second example, the angle A was proved to be 30°; not because A B was 8 feet, and B C 4 feet, but because A B was double of B C. If B C and A B had been 6 and 12, or 3 and 6, or 5 and 10 feet, the angle A would still be 30°, and the angle B 60°.

It is important to get this idea of *ratio* as *distinct from length* firmly and clearly into our minds; because, with that idea before us, Trigonometry becomes an easy study; but, without that idea, it is an impossible one.

CHAPTER II.

SINE, COSINE, AND TANGENT.

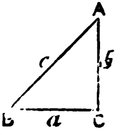
THERE being three sides to a triangle, and each of these having a determinate ratio to each of the other, we have, in all, six ratios.



Thus in the triangle A B C, we have the three sides, a, b, and c, and the ratios $\frac{a}{b}$, $\frac{a}{c}$, $\frac{b}{a}$, $\frac{b}{c}$, $\frac{c}{a}$, and $\frac{c}{b}$.

Each of these ratios has in Trigonometry a distinct name, and any one of the six, with one side, being given, is quite sufficient to determine the three angles and sides of a triangle having one of its angles a right angle: and for the present we confine our attention to right angled triangles, not because Trigonometry is in any sense confined to them, or applies in any especial way to them; but because, at first, the subject can be better understood by taking these alone into consideration.

The Sine.



The ratio $\frac{\textit{perpendicular}}{\textit{hypothenuse}} = \frac{\mathrm{A\,C}}{\mathrm{A\,B}} = \frac{b}{c}$ is called *the sine* of angle B.

By considering A C as the base, we have—

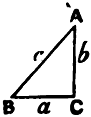
$$\frac{\mathrm{B\,C}}{\mathrm{A\,B}} = \frac{\textit{perpendicular}}{\textit{hypothenuse}} = \frac{a}{c} = \textit{sine} \text{ of angle A.}$$

The *hypothenuse* A B remaining the same, whether the angle B be increased or diminished, the *perpendicular* A C varies with B. When B is very small A C is very small, and the two increase together, until, when B is 90°, A C = A B, and $\frac{\textit{perpendicular}}{\textit{hypothenuse}} =$ sine of B = 1.

Therefore the ratio $\frac{\textit{perpendicular}}{\textit{hypothenuse}}$ increases from

0 to 1, and this is expressed, in Trigonometry, by saying—

sin B varies from 0 to 1.



If $B = 0°$, then $\frac{AC}{AB} = 0$.

If $B = 30°$, then $\frac{AC}{AB} = \frac{1}{2}$. (page 27.)

If $B = 45°$, then $\frac{AC}{AB} = \frac{1}{\sqrt{2}}$.

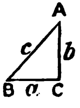
If $B = 60°$, then $\frac{AC}{AB} = \frac{\sqrt{3}}{2}$.

If $B = 90°$, then $\frac{AC}{AB} = 1$.

If, therefore, the ratio of $\frac{\textit{perpendicular}}{\textit{hypothenuse}}$ be between 0 and $\frac{1}{2}$, the angle B is less than 30°; if between $\frac{1}{2}$ and $\frac{1}{\sqrt{2}}$, the angle is between 30° and 45°; if between $\frac{1}{\sqrt{2}}$ and $\frac{\sqrt{3}}{2}$, the angle is between 45° and 60°; if between $\frac{\sqrt{3}}{2}$ and 1, the angle is between 60° and 90°.

We shall see hereafter that, by means of tables constructed for the purpose, we can find the exact angle corresponding to any given *sine*, and *vice versâ.*

The Cosine.



We saw that the perpendicular A C *varied with* the angle B, so long as the hypothenuse A B remained the same; but also, the base B C will be found to *vary inversely* with the angle B; *i. e.*, to *increase* as B *decreases*, and to *decrease* as B *increases*. Thus, when B is very small B C very nearly equals A B, and the larger the angle B becomes, the smaller is B C, until, when B = 90°, B C = 0.

Therefore, since B C decreases as angle B increases, and A B remains the same, the ratio $\frac{BC}{AB}$ will also decrease as B increases, but this ratio, $\frac{BC}{AB}$, is called the *cosine* of angle B; and thus in Trigonometry—

Cos. B varies *inversely* with B.

If $B = 0°$, then $\frac{BC}{AB} = 1$.

If $B = 30°$, then $\frac{BC}{AB} = \frac{\sqrt{3}}{2}$.

If $B = 45°$, then $\frac{BC}{AB} = \frac{1}{\sqrt{2}}$.

If $\quad B = 60°$, then $\frac{BC}{AB} = \frac{1}{2}$.

If $\quad B = 90°$, then $\frac{BC}{AB} = 0$.

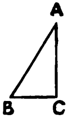
From this we see that if we have given us the value of $\frac{BC}{AB}$ we are able at once to form an *approximate* idea as to the magnitude of the angle B; and by the use of the tables already spoken of, we can find exactly the angle corresponding to any given value of $\frac{BC}{AB}$, and *vice versâ.*

By comparing the different values of the sine and cosine, as the angle increases from 0° and 90°, we shall get a better idea of their interdependence.

Magnitude of Angle.	Value of Sine.	Value of Cosine.
0°	$\frac{AC}{AB} = 0$	$\frac{BC}{AB} = 1$
30°	$\frac{AC}{AB} = \frac{1}{2}$	$\frac{BC}{AB} = \frac{\sqrt{3}}{2}$
45°	$\frac{AC}{AB} = \frac{1}{\sqrt{2}}$	$\frac{BC}{AB} = \frac{1}{\sqrt{2}}$
60°	$\frac{AC}{AB} = \frac{\sqrt{3}}{2}$	$\frac{BC}{AB} = \frac{1}{2}$
90°	$\frac{AC}{AB} = 1$	$\frac{BC}{AB} = 0$

As the *sine* increases, the *cosine* decreases, and *vice versá*. The greatest value of each is 1, and the least value is 0. If the *sine* and *cosine* be squared, the sum of the squares always = 1.

This last property may be better understood by a reference to the 47th Prop. of Euclid, Book I.



In a right angled triangle, the square of the hypothenuse A B equals the sum of the squares of A C and B C. This is equally true whether A C be equal to B C, or one be greater than the other. But if the hypothenuse continue the same, the perpendicular cannot be increased without the base being diminished, nor be decreased without the base becoming greater. Whatever be the ratio of the base to the perpendicular, the sum of their squares still remains the same, equal to the square of the hypothenuse.

Just so, regarding the perpendicular as representing the *sine*, and the base as representing the *cosine*, the squares of sine and cosine are, *together*, always the same, whatever be the ratio between them; since whatever alteration takes place in one, it is compensated for by a corresponding increase or decrease in the other.

Thus $AC^2 + BC^2 = AB^2$ (Euclid, I. 47.)

$$\therefore \frac{AC^2}{AB^2} + \frac{BC^2}{AB^2} = \frac{AC^2 + BC^2}{AB^2} = \frac{AB^2}{AB^2} = 1;$$

$$\therefore \textit{sine}^2 + \textit{cosine}^2 = 1.$$

This gives us an easy method of finding the sine of an angle from the cosine, or the cosine from the sine; for—

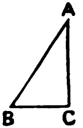
$$\text{sine} = \pm\sqrt{1 - \text{cosine}^2},$$

and also—

$$\text{cosine} = \pm\sqrt{1 - \text{sine}^2}.$$

The Tangent.

But besides the two ratios, $\frac{\textit{perpendicular}}{\textit{hypothenuse}}$ and $\frac{\textit{base}}{\textit{hypothenuse}}$, we have also the ratio $\frac{\textit{perpendicular}}{\textit{base}}$.



Thus $\frac{AC}{BC}$ is the *tangent* of B, and $\frac{BC}{AC}$ is the *tangent* of A.

When the angle B is 45°, then $AC = BC$, and $\frac{AC}{BC} = 1$; as the angle B increases (A B remaining the same), A C becomes greater and B C less; as the angle B decreases, A C becomes less and B C greater.

$\therefore$ when B greater than 45° $\frac{AC}{BC}$ greater than 1.

when B = 45° $\frac{AC}{BC} = 1$.

when B less than 45° $\frac{AC}{BC}$ less than 1.

If B = 0°, then $\frac{AC}{BC} = 0$.

If B = 30°, then $\frac{AC}{BC} = \frac{1}{\sqrt{3}}$.

If B = 45°, then $\frac{AC}{BC} = 1$.

If B = 60°, then $\frac{AC}{BC} = \sqrt{3}$.

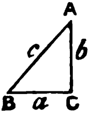
If B = 90°, then $\frac{AC}{BC} = \infty$.

This last is the symbol used to express infinity, or a magnitude without limit. When $B = 0$, the perpendicular also $= 0$, and the ratio $\frac{AC}{BC} = \frac{0}{BC} = 0$. But when the angle B is 90°, the perpendicular equal the hypothenuse, and the base becomes 0. Then $\frac{AC}{BC} = \frac{AC}{0}$, *i. e.*, the ratio is that of a line to nothing, and this ratio is infinite. This will be better understood by comparing with it what is said of the tangent on page 47.

Magnitude of Angle.	Value of Sine.	Value of Cosine.	Value of Tangent.
0°	0	1	0
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
45°	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
90°	1	0	∞

If these ratios be carefully examined, it will be noticed that, in every case, the value of the tangent is equal to the quotient of the sine divided by the cosine, *i. e.*, that $tangent = \frac{sine}{cosine}$.

This may be proved generally, for—



$$sine = \frac{b}{c},\quad cosine = \frac{a}{c}, \text{ and } tangent = \frac{b}{a};$$

$$\therefore \frac{sine}{cosine} = \frac{b}{c} \div \frac{a}{c} = \frac{b}{c} \times \frac{c}{a} = \frac{b}{a} = tangent.$$

Therefore, of the three ratios, *sine*, *cosine*, and *tangent*, from any two we can find the third, just as we saw (page 37) that we could find the sine from the cosine, or the cosine from the sine. Thus—

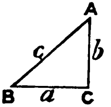
$$\textit{sine} \quad = \pm\sqrt{1 + \textit{cosine}^2} = \textit{tangent} \times \textit{cosine},$$

$$\textit{cosine} \quad = \pm\sqrt{1 + \textit{sine}^2} \quad = \frac{\textit{sine}}{\textit{tangent}},$$

$$\textit{tangent} = \frac{\textit{sine}}{\textit{cosine}}.$$

Examples.

In the right-angled triangle, A B C.



(1.) $b = 7$ and $c = 14$. Required a, A and B.
(2.) $b = 4$ and $a = 4$. „ c, A and B.
(3.) $a = 3$ and $c = 6$. „ b, A and B.
(4.) $b = 1{\cdot}74$ and $a = 3$. „ c, A and B.
(5.) $a = 3{\cdot}48$ and $c = 4$. „ b, A and B.
(6.) $a = 3{\cdot}48$ and $c = 400$. „ b, A and B.

(7.) A ladder 14·1 feet long made an angle of 45° with the side of a house. Required the remaining angle and sides of the triangle so formed.

(8.) A string 20 feet long, tied to the top of a tree, reached the ground at 10 feet from the root. Required the remaining side and angles of the triangle formed by the tree, string, and ground.

(9.) A boat crossing a stream 100 yards wide, made an angle of 30° with the direct route. Required the triangle formed by the lines of route and the opposite bank.

CHAPTER III.

The Relations between the Ratios.

The *sine* of 30° is $\frac{1}{2}$.

The *cosine* of 30° is $\frac{\sqrt{3}}{2}$.

The *cosine* of 60° is $\frac{1}{2}$.

The *sine* of 30° is $\frac{\sqrt{3}}{2}$.

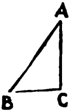
$$\therefore \textit{sine} \text{ of } 30^\circ = \textit{cosine} \text{ of } 60^\circ;$$
$$\text{also } \textit{cosine} \text{ of } 30^\circ = \textit{sine} \text{ of } 60^\circ.$$

Likewise *sine* of $45^\circ = \frac{1}{\sqrt{2}} =$ *cosine* of 45°.

But $30^\circ + 60^\circ = 90^\circ$, also $45^\circ + 45^\circ = 90^\circ$.

From these we may infer that, generally, the sine of any angle = cosine of the complement of the angle; *i. e.*, that the *sine* of one *acute* angle of a right-angled triangle = the *cosine* of the other, and *vice versâ.*

That this is true may be proved thus—



The *sine* of B is $\frac{A\,C}{A\,B}$, and the *cosine* is $\frac{B\,C}{A\,B}$.

The *cosine* of A is $\frac{A\,C}{A\,B}$, and the *sine* is $\frac{B\,C}{A\,B}$.

$\therefore$ *sine* of A $=$ *cosine* $(90^\circ - \mathrm{A})$;
and *cosine* of A $=$ *sine* $(90^\circ - \mathrm{A})$.

Just as the *sine* of A is called the *cosine* of B, and the *sine* of B is called the *cosine* of A, so the *tangent* of A is called the *cotangent* of B, and the *tangent* of B is called the *cotangent* of A.

Also, just as $\textit{tangent} = \frac{\textit{sine}}{\textit{cosine}}$, so $\textit{cotangent} = \frac{\textit{cosine}}{\textit{sine}}$.

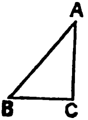
We saw that the *sine* and *cosine* were connected by the equation $\textit{sine}^2 + \textit{cosine}^2 = 1$, and that one could always be found from the other by means of this equation.

So the *tangent* and *cotangent* are connected by the equation $\textit{tangent} = \frac{1}{\textit{cotangent}}$, or $\textit{tangent} \times \textit{cotangent} = 1$; from which also $\textit{cotangent} = \frac{1}{\textit{tangent}}$.

But besides the four ratios already mentioned, namely $\frac{A\,C}{A\,B}$, $\frac{B\,C}{A\,B}$, $\frac{A\,C}{B\,C}$, and $\frac{B\,C}{A\,C}$, we have the two $\frac{A\,B}{A\,C}$ and $\frac{A\,B}{B\,C}$, or $\frac{\textit{hypothenuse}}{\textit{perpendicular}}$ and $\frac{\textit{hypothenuse}}{\textit{base}}$.

The ratio $\frac{A\,B}{B\,C}$ is called the *secant* of B, while $\frac{A\,B}{A\,C}$ is the *secant* of A. Speaking generally, the ratio $\frac{\textit{hypothenuse}}{\textit{base}}$ is the *secant* of the base acute angle of a right-angled triangle.

Just as in the case of the *sine* and *cosine*, the *secant* of B is the *cosecant* of A, and the *secant* of A is the *cosecant* of B.



The six ratios may be grouped together thus:—

Of angle B, or (90°—A).		Of angle A, or (90°—B).
sine	$= \frac{b}{c} =$	*cosine*
cosine	$= \frac{a}{c} =$	*sine*
tangent	$= \frac{b}{a} =$	*cotangent*
cotangent	$= \frac{a}{b} =$	*tangent*
secant	$= \frac{c}{a} =$	*cosine*
cosecant	$= \frac{c}{b} =$	*sine*

It will be seen also, that—

$$\text{sine} = \frac{1}{\text{cosecant}}, \qquad \text{cosine} = \frac{1}{\text{secant}};$$

$$\text{tangent} = \frac{1}{\text{cotangent}}, \qquad \text{cotangent} = \frac{1}{\text{tangent}};$$

$$\text{secant} = \frac{1}{\text{cosine}}, \qquad \text{cosecant} = \frac{1}{\text{sine}}.$$

We have seen that the *sine* may be expressed in the terms of the *cosine*, that the *tangent* may be written as $\frac{\text{sine}}{\text{cosine}}$, and that the *secant* may be written $\frac{1}{\text{cosine}}$.

We can now consider this more fully, and a little consideration will enable us to express *any one ratio* in the terms of *any other*.

It will be more convenient, however, in these formulæ to use the ordinary contracted forms of the names, which are—

sin for sine.
tan for tangent.
sec for secant.
cos for cosine.
cotan for cotangent.
cosec for cosecant.

From the fundamental formula—

$$sin^2 + cos^2 = 1,$$

we may obtain a complete set of formulæ, thus—(see page 45).

Each vertical column gives the respective values of the six ratios, expressed in the terms of one, while each horizontal column gives the various expressions for the value of one ratio. This table gives the value of each of the six ratios in terms of each of the others. This is of very great *practical utility* in proving propositions, and in calculations respecting the sides and angles of triangles.

Examples.

(1.) Sine $= \frac{1}{3}$
(2.) Cosine $= \frac{1}{2}$
(3.) Tangent $= 4$
(4.) Sine $= \frac{1}{8}$
(5.) Cosecant $= 2$
(6.) Cotangent $= \frac{1}{4}$
(7.) Secant $= 2$
(8.) Sine $= -\frac{1}{2}$
(9.) Cosine $= -\frac{1}{4}$
(10.) Tangent $= -\frac{1}{2}$
(11.) Secant $= -7$
(12.) Cosecant $= -6$

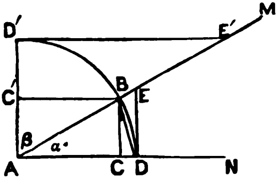
Find the values of the remaining ratios.

These negative ratios are given to accustom the

	Sine.	*Cosecant.*	*Cosine.*	*Secant.*	*Tungent.*	*Cotangent.*
Sine . .	1	$\frac{1}{cosec}$	$\sqrt{1-cos^2}$	$\frac{\sqrt{sec^2-1}}{sec}$	$\frac{tan}{\sqrt{tan^2+1}}$	$\frac{1}{\sqrt{1+cot^2}}$
Cosecant .	$\frac{1}{sin}$	1	$\frac{1}{\sqrt{1-cos^2}}$	$\frac{sec}{\sqrt{sec^2-1}}$	$\frac{\sqrt{tan^2+1}}{tan}$	$\sqrt{1+cot^2}$
Cosine .	$\sqrt{1-sin^2}$	$\frac{\sqrt{cosec^2-1}}{cosec}$	1	$\frac{1}{sec}$	$\frac{1}{\sqrt{1+tan^2}}$	$\frac{cot}{\sqrt{1+cot^2}}$
Secant .	$\frac{1}{\sqrt{1-sin^2}}$	$\frac{cosec}{\sqrt{cosec^2-1}}$	$\frac{1}{cos}$	1	$\sqrt{1+tan^2}$	$\frac{\sqrt{1+cot^2}}{cot}$
Tungent .	$\frac{sin}{\sqrt{1-sin^2}}$	$\frac{1}{\sqrt{cosec^2-1}}$	$\frac{\sqrt{1-cos^2}}{cos}$	$\sqrt{sec^2-1}$	1	$\frac{1}{cot}$
Cotangent	$\frac{\sqrt{1-sin^2}}{sin}$	$\sqrt{cosec^2-1}$	$\frac{cos}{\sqrt{1-cos^2}}$	$\frac{1}{\sqrt{sec^2-1}}$	$\frac{1}{tan}$	1

mind to them, and to give facility in calculating them, as well as the positive values. The utility of this will be shown hereafter. Meantime there need be no difficulty in working them.

CHAPTER IV.



Let two straight lines A M and A N make any angle, as M A N.

Take any point, B, in A M, and draw B C perpendicular to A N.

From A as centre, with radius A B describe a circle, of which D′ B D will be a quadrant, D′ A being drawn perpendicular to A N.

Through D draw D E, a tangent to the circle.

If necessary, produce A M to cut D E, and draw B C′ and D′ E′ parallel to A N.

Then—

$$\left.\begin{array}{l}BC \text{ is the } \textit{sine}\\ DE \text{ is the } \textit{tangent}\\ AE \text{ is the } \textit{secant}\end{array}\right\}\text{ of angle } a.$$

Here we speak of the *sine*, *tangent*, and *secant*, not as *ratios*, but as *lines*.

It will be seen that the *sine* B C increases from 0 to the length of the radius of the circle, as the angle A increases from 0° to 90°: that the *tangent* increases from 0 to ∞: and that the *secant* increases from length of radius to ∞.

These facts are precisely the same as have been already proved for the *ratios* (page 39), though apparently somewhat different.

But the *lengths* of the lines B C, D E, and A E, depend entirely upon the distance of B from A. If we take B nearer to A we shall make the *sine*, *tangent*, and *secant*, less in length; and if we take it farther from A we make them longer. Therefore, we can have a *sine*, *tangent*, or *secant*, of any length we please, whatever be the angle A.

But if we consider, not the absolute length of these lines, but their *ratio to the radius* of the circle, we shall find this ratio to be the same whatever be the distance of B from A.

We shall also find the values of these ratios to be identical with those already given (page 39).

Thus, reckoning the radius as 1 (whether it be 1 inch, 1 foot, 1 yard, &c.), we shall have, as the angle increases from 0° to 90°;

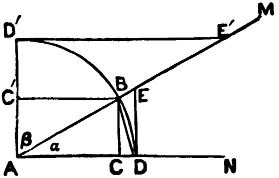
B C = *sine* of A, increases from 0 to 1,
D E = *tangent*, increases from 0 to ∞,
A E = *secant*, increases from 1 to ∞;

which values are identical with those on page 39.

But the utility of *these sines*, *tangents*, and *secants*, is dependent upon a knowledge of the length of the radius A B; *i. e.*, it is no use to know the length of B C, D E, or A E, unless we also know what distance B is from the angular point A.

But—
B C : A B : : *sine** : *radius* = *sine* † of A;
D E : A D : : *tangent** : *radius* : : B C : A C = *tangent.*†
and—
A E : A D : : *secant** : *radius* : : A B : A C = *secant.*†

From this we get the three *ratios* $\frac{BC}{AB}$, $\frac{BC}{AC}$, and $\frac{AB}{AC}$, which we use, in modern Trigonometry, *instead of the lines* D E, B C, and A E divided respectively by the radius; and which have the advantage of being quite independent of the length of A B.



If we take the angle (90° − α), marked β, we may consider A D′ the base, and then B C′ becomes the *sine*, D′ E′ the *tangent*, and A E′ the *secant* of this angle. This angle, β, is the complement, or co-angle, of α, and the *sine*, *tangent*, and *secant* of it become, respectively, the *cosine*, *cotangent*, and *cosecant* of α.

Also the *sine*, *tangent*, and *secant* of angle α are respectively the *cosine*, *cotangent*, and *cosecant* of angle β.

But, as we have seen, we must have either the *length* of the radius A B, or the *ratio* of the *sine*, *tangent*, or *secant* to the radius.

* Meaning the *lines* so called.
† Meaning the *ratios* so called.

Then $BC' : AB :: AC : AB$; For $AC = BC'$.

$$i.\,e.,\ \textit{sine of } \beta = \frac{BC'}{AB} = \textit{cosine of } \alpha.$$

Also $D'E' : AD' :: BC' : AC'$;

$$i.\,e.,\ \textit{tangent of } \beta = \frac{BC'}{AC'} = \textit{cotangent of } \alpha.$$

Also $AE' : AD' :: AB : AC'$

$$i.\,e.,\ \textit{secant of } \beta = \frac{AB}{AC'} = \textit{cosecant of } \alpha.$$

These facts may be summed up thus:—

$$\left.\begin{array}{l} BC = \textit{sine} \\ DE = \textit{tangent} \\ AE = \textit{secant} \end{array}\right\} \text{of angle } \alpha.$$

Equal, respectively, to *cosine*, *cotangent*, and *cosecant* of angle β.

$$\left.\begin{array}{l} BC' = \textit{sine} \\ D'E' = \textit{tangent} \\ AE' = \textit{secant} \end{array}\right\} \text{of angle } \beta.$$

Equal, respectively, to *cosine*, *cotangent*, and *cosecant* of angle α.

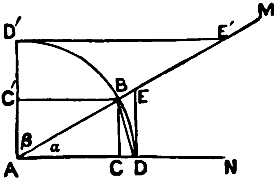
Since the *length* of these lines depends entirely upon the distance of B from A, we cannot, from merely knowing this length, find anything concerning the magnitude of the angles α or β; but if we know also the length of the radius, we have sufficient data to enable us to find the magnitude of both angles.

Then we have—

$$\left.\begin{array}{l} \dfrac{BC}{\textit{radius}} = \textit{sine} \\[2ex] \dfrac{DE}{\textit{radius}} = \textit{tangent} \\[2ex] \dfrac{AE}{\textit{radius}} = \textit{secant} \end{array}\right\} \text{of angle } \alpha.$$

Also—

$$\left.\begin{aligned}\frac{\mathrm{B\,C'}}{\textit{radius}} &= \textit{sine}\\ \frac{\mathrm{D'\,E'}}{\textit{radius}} &= \textit{tangent}\\ \frac{\mathrm{A\,E'}}{\textit{radius}} &= \textit{secant}\end{aligned}\right\}\text{ of angle } \beta.$$



From these, remembering that $\mathrm{A\,D} = \mathrm{A\,B} = \mathrm{A\,D'}$ (all being radii of the circle), and that the triangles A B C, A D E, A B C′, and A D′ E′, are all equiangular, and therefore their sides proportional (Euclid, VI.), we get the convenient formulæ:—

$$\textit{Sine } \alpha = \frac{\mathrm{B\,C}}{\mathrm{A\,B}} = \frac{1}{\textit{cosec } \alpha} = \textit{cosine } \beta = \frac{1}{\textit{secant } \beta}.$$

$$\textit{Cosine } \alpha = \frac{\mathrm{A\,C}}{\mathrm{A\,B}} = \frac{1}{\textit{secant } \alpha} = \textit{sine } \beta = \frac{1}{\textit{cosec } \beta}.$$

$$\textit{Secant } \alpha = \frac{\mathrm{A\,B}}{\mathrm{A\,C}} = \frac{1}{\textit{cosine } \alpha} = \textit{cosec } \beta = \frac{1}{\textit{sine } \beta}.$$

$$\textit{Cosecant } \alpha = \frac{\mathrm{A\,B}}{\mathrm{B\,C}} = \frac{1}{\textit{sine } \alpha} = \textit{secant } \beta = \frac{1}{\textit{cosine } \beta}.$$

$$\textit{Tangent } \alpha = \frac{\mathrm{B\,C}}{\mathrm{A\,C}} = \frac{1}{\textit{cotan } \alpha} = \textit{cotan } \beta = \frac{1}{\textit{tan } \beta}.$$

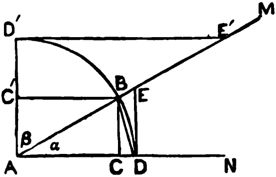
$$\textit{Cotangent}\ \alpha = \frac{AC}{BC} = \frac{1}{\tan\alpha} = \tan\beta = \frac{1}{\cot\!an\ \beta}.$$

$$\textit{Chord}\ \alpha = \frac{BD}{AD}.$$

$$\textit{Versed sine}\ \alpha = \frac{CD}{AB} = 1 - \textit{cosine}\ \alpha = 1 - \sin\beta.$$

These two last ratios, the *chord* and *versed sine*, are less used than the others, but are occasionally very useful.

Examples.



(1.) A B = 4, A E = 5, B C = 1.
Find A C, A D, and D E.

(2.) A B = 5, C D = 1.
Find A C, B E, B C, and D E.

(3.) A C = D C.
Find the value of $\frac{AB}{AE}$, and of $\frac{BC}{DE}$.

(4.) Find the magnitude of angle β when B C = D E.

(5.) Find the magnitudes of α and β when D′ E′ = D E.

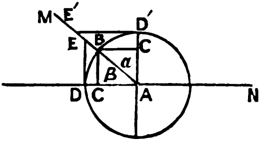
(6.) Find the magnitude of angle A B C when A E′ = 2 A D′.

(7.) A B = B D. Prove β = 30°.

(8.) A B = 5. Find A C, A E, and D E, if β = 30°.

CHAPTER V.

OF THE RATIOS OF ANGLES GREATER THAN 90°.



THERE is no reason why the angles α and β may not be greater than 90°, *i.e.*, be obtuse angles. If so, will the definitions we have found true for acute angles remain true? or, shall we have to extend our definitions as we extend the angles? We shall find that our definitions remain true; but we shall find it necessary to have a more comprehensive view of both angles and ratios.

If we draw an obtuse angle, M A N, and proceed to draw sine, tangent, and secant, our first difficulty will be, that we cannot draw B C perpendicular to A N. Therefore, we must extend our idea of a perpendicular to a given line to include the case where it is necessary to make the given line longer. If we produce the line N A beyond A, we can then draw B C perpendicular to it, and this will be the sine of the angle α. The perpendicular, D E, will be the tangent, and the line, A E, the secant. These lines will be *represented*, as before, by the ratios $\frac{BC}{BA}$ = *sine*, $\frac{BC}{CA}$ = *tangent*, and $\frac{AB}{AC}$ = *secant*.

There are now two points to consider:

1st. That the sine, tangent, and secant of an angle increase with the angle as it increases from 0 to 90°; but that, as the angle *increases* from 90° to 180°, the sine, tangent, and secant *decrease* in magnitude.

2nd. That the sine, tangent, and secant of an angle greater than 90°, are precisely the same as those of the *supplement* of that angle, *i.e.*, the sine, tangent, and secant of α = those of β, which is $(180^\circ - a)$. By a is meant the angle M A N.

As to the first point:

We see that the sine increases from 0 to 1, as the angle increases from 0 to 90°, and then decreases from 1 to 0 again.

The tangent increases from 0 to ∞, and then decreases from ∞ to 0 again.

The secant increases from 1 to ∞, and then decreases from ∞ to 1 again.

Therefore, the sine, tangent, and secant of 0° are the same as those of 180°; and this is to be expected, since an angle of 0° and one of 180°, are equal in one respect, *i.e.*, they are both represented by a straight line.

In the same way, and for the same reason, the cosine decreases from 1 to 0, and then increases from 0 to 1; the cotangent decreases from ∞ to 0, and then increases from 0 to ∞; and the cosecant decreases from ∞ to 1, and then increases from 1 to ∞.

These are *represented* by the ratios $\frac{\text{A C}}{\text{A B}}$ = *cosine*, $\frac{\text{A C}}{\text{B C}}$ = *cotangent*, and $\frac{\text{A B}}{\text{B C}}$ = *cosecant*.

As to the second point, the equality of the ratios belonging to two different angles:

Since the ratios of α and of $(180^\circ - \alpha)$ are precisely of the same magnitude, it may be asked, how can we tell which is meant? The answer requires a knowledge of the use of the *minus sign* in Algebraic Geometry, which, if not already possessed, may be obtained from a perusal of the chapters on that subject in the treatise on Euclid, vol. II. page 173.

The distance A C, when measured to the right of A, is positive; but when measured to the left of A is called negative and is expressed by $-\mathrm{A\,C}$.

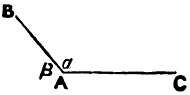
In the first quadrant (*i.e.*, when the angle is less than 90°), all the lines and ratios are positive; but in the second quadrant (when the angle is between 90° and 180°), the line A C is $-$ A C, and this affects every ratio into which that quantity enters, *i.e.*, all but the *sine* and *cosecant*.

Thus, in the second quadrant,—

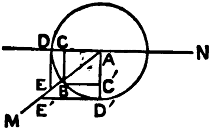
$$\begin{array}{ll|ll}
\textit{sine} & = \dfrac{\mathrm{BC}}{\mathrm{AB}} = +\dfrac{\mathrm{BC}}{\mathrm{AB}} & \textit{cosine} & = \dfrac{-\mathrm{AC}}{\mathrm{AB}} = -\dfrac{\mathrm{AC}}{\mathrm{AB}} \\
\textit{tangent} & = \dfrac{\mathrm{BC}}{-\mathrm{AC}} = -\dfrac{\mathrm{BC}}{\mathrm{AC}} & \textit{cotang} & = \dfrac{-\mathrm{AC}}{\mathrm{BC}} = -\dfrac{\mathrm{AC}}{\mathrm{BC}} \\
\textit{secant} & = \dfrac{\mathrm{AB}}{-\mathrm{AC}} = -\dfrac{\mathrm{AB}}{\mathrm{AC}} & \textit{cosec} & = \dfrac{\mathrm{AB}}{\mathrm{BC}} = +\dfrac{\mathrm{AB}}{\mathrm{BC}}
\end{array}$$

We have now somewhat extended our notions of the ratios of Trigonometry; but we must extend still further, not only our ideas of the magnitudes and signs of the ratios, but of the magnitudes of the angles themselves. Thus, we have already spoken of angles of 0° and of 180°, both represented by straight lines, and neither of which is, in the ordinary sense of the word, an angle at all.

We must now consider angles *greater than* 180°, *i.e.*, greater than two right angles.



Thus we speak of angle B A C, as being an obtuse angle—greater than 90°, but less than 180°; but we must now consider the *second angle formed by the same lines, i.e.*, the angle β, also written B A C, but greater than 180°.



Our definitions will still hold good. Taking any point B, in A M, and describing the circle with radius A B round A as a centre, we have, as before:

B C = *sine* expressed by the ratio $\frac{BC}{AB}$

D E = *tangent* ,, ,, ,, $\frac{BC}{AC}$

A E = *secant* ,, ,, ,, $\frac{AB}{AC}$;

also—

B C′ = *cosine* expressed by the ratio $\frac{AC}{AB}$

D′ E′ = *cotang* ,, ,, ,, $\frac{AC}{BC}$

A E′ = *cosecant* ,, ,, ,, $\frac{AB}{BC}$.

Our angle now extends over the first, second, and part of the third quadrants, *i.e.*, the line A M is in the third quadrant, counting from A N. The *magnitude* of all the ratios are precisely the same as if the angle were 180° less than it is, *i.e.*, the ratios of the angle M A N are *the same in magnitude* as the ratios of the angle D A M. Therefore, we may extend our theorem of page 53, by saying that all ratios of angle $a =$ ratios of $180° + a =$ all ratios of $180° - a$, which may be written, ratios of $a =$ ratios of $180° \pm a$.

But, though the *magnitudes* are the same, there will be found some difference in the *signs*, just as we found some difference between the signs of ratios in the first and second quadrant respectively.

In the second quadrant, A C was found to be − A C, because measured to the left of A. In like manner B C, which is *positive* when measured upwards from the line on which A stands, is *negative* when below that line.

Therefore:

sine = B C represented by $\frac{-BC}{AB} = -\frac{BC}{AB}$

tangent = D E „ „ $\frac{-BC}{-AC} = +\frac{BC}{AC}$

secant = A E „ „ $\frac{AB}{-AC} = -\frac{AB}{AC}$;

also,—

cosine = B C′ represented by $\frac{-AC}{AB} = -\frac{AC}{AB}$

cotang = D′ E′ „ „ $\frac{-AC}{-BC} = +\frac{AC}{BC}$

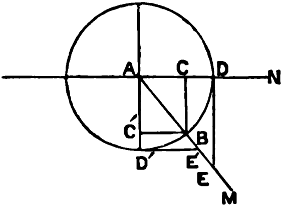
cosecant = A E′ „ „ $\frac{AB}{-BC} = -\frac{AB}{BC}$.

Therefore, in the third quadrant, all the ratios are −, except the tangent and cotangent.

Notice that, in the third quadrant, *every* ratio has at least *one minus* sign, but that the tangent and cotangent have each *two*, and this gives a positive value to the ratios.

Finally, our angle may be greater than 270° but less than 360°, *i.e.*, the line A M may be in the fourth quadrant.

The definitions will still hold good, but the magnitudes will be the same as if the angles were 270° less than they are, and the *signs* will be still different:



sine = B C represented by $\frac{-BC}{AB} = -\frac{BC}{AB}$

tangent = D E ” ” $\frac{-BC}{AC} = -\frac{BC}{AC}$

secant = A E ” ” $\frac{AB}{AC} = +\frac{AB}{AC}$;

also,—

cosine = B C′ represented by $\frac{AC}{AB} = +\frac{AC}{AB}$

cotang = D′ E′ ” ” $\frac{AC}{-BC} = -\frac{AC}{BC}$

cosecant = A E′ ” ” $\frac{AB}{-BC} = -\frac{AB}{BC}$.

Therefore, in the fourth quadrant, the ratios are all −, except the *secant* and *cosine*.

The radius A B is *always* considered *positive*.

These changes of *signs* may be summarized thus:

—	*Sine.*	*Cosecant.*	*Secant.*	*Cosine.*	*Tangent.*	*Cotangent.*
1st quadrant, angle between 0° & 90°	+	+	+	+	+	+
2nd „ „ 90° & 180°	+	+	−	−	−	−
3rd „ „ 180° & 270°	−	−	−	−	+	+
4th „ „ 270° & 360°	−	−	+	+	−	−

The *limits* of changes in *magnitude* are shown thus:

	0°	90°	180°	270°	360°
Sine.	0	1	0	−1	0
Tangent . . .	0	∞	0	∞	0
Secant	1	∞	−1	∞	1
Cosine . . .	1	0	−1	0	1
Cotangent . .	∞	0	∞	0	∞
Cosecant . . .	∞	1	∞	−1	∞

From these two tables we gather that the *sign* of a ratio alters when, and only when, its magnitude passes through ∞ or 0. Also, that the magnitudes of the ratios have fixed limits.

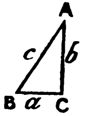
Thus:

sine . . .	always between 1 and − 1,
cosine . .	*i. e.*, never *greater* than 1,
secant . .	always between 1 and ∞,
cosecant. .	*i. e.*, never *less* than 1,
tangent. .	may be any magnitude.
cotangent .	

We saw that the ratios of 0° and those of 180° were equal in magnitude; so also are those of 0° and 360°. In fact, these may be considered identical, since the lines A M and A N coincide.

We might now extend our notions of an angle still further, and consider angles of greater magnitude than 360°; but the ratios of any such angle are identical, both in sign and magnitude, with those of 360° less, *i.e.*, ratios of any angle $360° + a$ = ratios of a in both magnitude and sign. Therefore, we need not discuss these larger angles here, though it is important to remember that, in Trigonometry, there is *no limit* to the magnitude of angles.

Examples.



(1.) If $a=3$, $b=-4$. Find the values of A and B.

(2.) If $a=-3$, $b=-4$. „ „

(3.) The sine of an angle is $\frac{1}{2}$, the cosine $-\frac{\sqrt{3}}{2}$.
Find magnitude of angle and values of remaining ratios.

(4.) Secant $= 2$, tangent $= -\sqrt{3}$.
Find the other ratios and the magnitude of the angle.

(5.) The cotangent $= -2$ cosine.
Find the other ratios and the angle.

(6.) The sine $= \dfrac{\text{tangent}}{2}$.
Find the other ratios and the angle.

CHAPTER VI.

SOLUTION OF TRIANGLES.

We have already (page 30) seen how it is possible to find the remaining parts of a triangle when three of the six parts are given. We can now discuss more fully this subject, to which the name of 'Solution of Triangles' is generally given.

We may use, in the course of our solutions, any of the facts or formulæ we have already proved, and there are some others, of very considerable use, which we will now consider.

Theorem.—Any two sides of any triangle are in the same ratio to each other as the sines of the angles respectively opposed to them, *i.e.* :—

$$\left.\begin{array}{l} \sin \mathrm{A} : \sin \mathrm{B} :: a : b \\ \sin \mathrm{A} : \sin \mathrm{C} :: a : c \\ \sin \mathrm{B} : \sin \mathrm{C} :: b : c \end{array}\right\}$$

A, B, C, express the respective angles, and a, b, c, the sides respectively opposite to them; d is the perpendicular from A on the base.

For $\frac{d}{c} = \sin A \;\therefore d = c \sin A,$

also, $\frac{d}{a} = \sin C \;\therefore d = a \sin C,$

$$\therefore c \sin A = a \sin C,$$

$$\therefore \frac{\sin A}{\sin C} = \frac{a}{c}.$$

It should be borne in mind that this is *not* the same as $\frac{A}{C} = \frac{a}{c}$.

Likewise, $\frac{\sin A}{\sin B} = \frac{a}{b}$, and $\frac{\sin B}{\sin C} = \frac{b}{c}$.

Theorem.—*Cos* $c = \frac{a^2 + b^2 - c^2}{2\,a\,b}$,

$\left(\text{also, } \cos a = \frac{b^2 + c^2 - a^2}{2\,b\,c}, \text{and } \cos b = \frac{a^2 + c^2 - b^2}{2\,a\,c}\right).$

For $c^2 = a^2 + b^2 - 2\,a\,.\,CD$ (Eucl., II. 13),

$$\left(\text{but } \frac{CD}{b} = \cos C \;\therefore CD = b\,.\cos C,\right)$$

$$\therefore c^2 = a^2 + b^2 - 2\,a\,.\,b\,.\cos C,$$

$$\therefore 2\,a\,b \cos C = a^2 + b^2 - c^2,$$

and $$\cos C = \frac{a^2 + b^2 - c^2}{2\,a\,b}.$$

We are now in a position to solve some simple cases of triangles.

Example.—$b = 12$ feet, $a = 6$ feet, and $B = 90°$. Required the values of A, C, and c.

$$b : a :: \sin 90° : \sin A;$$

$$\therefore b \sin A = a\,.\sin 90° \quad (\sin 90° = 1),$$

$$\therefore \quad \sin A = \frac{a}{b} = \frac{6}{12} = \frac{1}{2},$$

$$\therefore \quad A = 30^\circ. \qquad \because \sin 30^\circ = \frac{1}{2}.$$

Also, $C = 180^\circ - (A + B) = 180^\circ - 120^\circ = 60^\circ$;

and $a : c :: \sin A : \sin C$,

$$\therefore a \, . \, \frac{\sqrt{3}}{2} = c \, . \, \frac{1}{2} \qquad \left(\sin 60^\circ = \frac{\sqrt{3}}{2}\right),$$

$$\therefore c = a \, . \, \frac{\sqrt{3}}{2} \div \frac{1}{2}$$

$$= 6 \, . \, \sqrt{3} = 6 \times 1{\cdot}73 + = 10{\cdot}38 +,$$

$$\therefore a = 6, \; b = 12, \text{ and } c = 10{\cdot}38 + ;$$

also, $A = 30^\circ$, $B = 90^\circ$, and $C = 60^\circ$;
and the triangle is completely known.

Example.—$A = 60^\circ$, $B = 30^\circ$, $c = 10$ yards.
Required the values of C, a, and b.

$$C = 180^\circ - (A + B) = 180^\circ - 90^\circ = 90^\circ.$$

Also, $a : c :: \sin A : \sin C$,

$$\therefore a = \frac{c \, . \sin A}{\sin C},$$

$$= \frac{10 \times \sqrt{3}}{2 \times 1},$$

$$= 5\sqrt{3}.$$

And $b : c :: \sin B : \sin C$,

$$\therefore b = \frac{c \, . \sin B}{\sin C},$$

$$= \frac{10 \times 1}{2 \times 1},$$

$$= 5.$$

Therefore $C = 90^\circ$, $a = 5\sqrt{3}$, $b = 5$.

Example.—$A = 90°$, $B = 30°$, and $a = 4$ feet.

Required the values of C, b, and c.

$$C = 180° - (90° + 30°) = 60°;$$

also $$a : b :: 1 : \frac{1}{2},$$

$$\therefore b = 4 \times \frac{1}{2} = 2;$$

and $$a : c :: 1 : \frac{\sqrt{3}}{2},$$

$$\therefore c = 4 \times \frac{\sqrt{3}}{2} = 2\sqrt{3} = 3 \cdot 46 +,$$

$$\therefore A = 90°, B = 30°, \text{ and } C = 60°;$$

also $a = 4$ feet, $b = 2$ feet, and $c = 3 \cdot 46$ feet.

Example.—$a = 5$ feet, $b = 5$ feet, and $C = 90°$,

$$a : b :: \sin A : \sin B;$$

but $$a = b \therefore A = B, \text{ and each} = 45°;$$

$$\therefore c : a :: 1 : \frac{1}{\sqrt{2}};$$

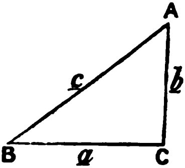
$$\therefore c = \frac{5}{\frac{1}{\sqrt{2}}} = 5\sqrt{2} = 7 \cdot 05 +.$$

Therefore $A = 45°$, $B = 45°$, and $C = 90°$;

also $a = 5$ feet, $b = 5$ feet, and $c = 7 \cdot 05$ feet.

These four examples are simple, but at the same time *general*; *i.e.*, they show the *general method* of procedure, with the exception of the last one, which is special, inasmuch as two of the angles are equal. The *general* treatment of the problem of solving a triangle of which two sides and the included angle are given, will be found at page 105. It requires more special knowledge than we have yet attained.

Examples.



Right-angled triangles.

(1.) $c = 4$, $b = 2$. Find a, A, and B.
(2.) $a = 17\cdot4$, $c = 20$. Find b, A, and B.
(3.) $b = 2$, $c = \sqrt{8}$. Find a, A, and B.

CHAPTER VII.

Ratios of Compound Angles.

It is sometimes necessary to express the sine, tangent, &c., of an angle, in the terms of the sines, tangents, &c., of other angles. Thus the angle of 45° may be considered not only as 45°, but as (30°+15°), (25°+20°), &c., and also as (60°−15°), (90°−45°), &c.

Suppose we want to express the *sine* of 60° in terms of 30° and 90° (for 60° = 90° − 30°) we shall find that it is possible to do so, and that the formula will be—

$$\sin 90^\circ . \cos 30^\circ - \cos 90^\circ . \sin 30^\circ;$$

i.e., the product of *cos* 90° and *sin* 30° subtracted from the product of *sin* 90° and *cos* 30° will leave *sin* 60°.

Substituting the values, we have—

$$\begin{aligned} \sin 60^\circ &= 1 \times \frac{\sqrt{3}}{2} - 0 \times \frac{1}{2}, \\ &= \frac{\sqrt{3}}{2} - 0, \\ &= \frac{\sqrt{3}}{2}, \text{ which is the value of } \sin 60^\circ. \end{aligned}$$

To take another instance:—

$$90^\circ = 150^\circ - 60^\circ.$$

If we wish to express *sin* 90° in terms of *sin* and *cos* of 150° and 60°, we have as before—

$$\begin{aligned} \sin 90^\circ &= \sin 150^\circ \,.\, \cos 60^\circ - \cos 150^\circ \,.\, \sin 60^\circ, \\ &= \left(\frac{1}{2} \cdot \frac{1}{2}\right) - \left(-\frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2}\right), \\ &= \frac{1}{4} + \frac{3}{4}, \\ &= 1, \text{ which is the } sine \text{ of } 90^\circ. \end{aligned}$$

Also $60^\circ = 150^\circ - 90^\circ$, and

$$\begin{aligned} \sin 60^\circ &= \sin 150^\circ \,.\, \cos 90^\circ - \sin 90^\circ \,.\, \cos 150^\circ, \\ &= \left(\frac{1}{2} \,.\, 0\right) - \left(1 \,.\, -\frac{\sqrt{3}}{2}\right), \\ &= 0 - \left(-\frac{\sqrt{3}}{2}\right), \\ &= \frac{\sqrt{3}}{2}, \text{ which is the } sine \text{ of } 60^\circ. \end{aligned}$$

But, also $90^\circ = 60^\circ + 30^\circ$, $150^\circ = 90^\circ + 60^\circ$, &c.; *i.e.*, we may require to express one angle as the *sum of two others*. Then we shall find the formula—

$$\sin 150^\circ = \sin 90^\circ \,.\, \cos 60^\circ + \sin 60^\circ \,.\, \cos 90^\circ,$$

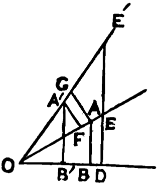
i.e., we take the *sum* of the products instead of their difference.

For example:—

$$\begin{aligned} \sin 150^\circ &= \sin 90^\circ \,.\, \cos 60^\circ + \sin 60^\circ \,.\, \cos 90^\circ \\ &= 1 \,.\, \frac{1}{2} + \frac{\sqrt{3}}{2} \,.\, 0, \\ &= \frac{1}{2}, \text{ which is the } \textit{sine} \text{ of } 150^\circ. \end{aligned}$$

Also—

$$\begin{aligned} \sin 90^\circ &= \sin 60^\circ \,.\, \cos 30^\circ + \sin 30^\circ \,.\, \cos 60^\circ, \\ &= \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{1}{2} \cdot \frac{1}{2}, \\ &= \frac{3}{4} + \frac{1}{4}, \\ &= 1, \text{ which is the } \textit{sine} \text{ of } 90^\circ. \end{aligned}$$



Let A O B be any angle, and A O A any other angle; then the whole angle A′ O B is the *sum* of the other two.

Now, $$\left.\begin{array}{l} AB = \sin \\ OB = \cos \\ DE = \tan \\ OE = \sec \end{array}\right\} \text{ of angle } a.$$

Also, $$\left.\begin{array}{l} A'F = \sin \\ OF = \cos \\ AG = \tan \\ OG = \sec \end{array}\right\} \text{ of angle } \beta.$$

Lastly, $$\left.\begin{array}{l} A'B' = \sin \\ OB' = \cos \\ DE' = \tan \\ OF' = \sec \end{array}\right\} \text{ of angle } a + \beta.$$

Then our theorem is that—

$$A'B' = AB \cdot OF + A'F \cdot OB;$$

also that—

$$AB = A'B' \cdot OF - A'F \cdot OB;$$

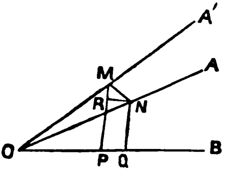
and that—

$$A'F = A'B' \cdot OB - AB \cdot OB'.$$

But the proof of this is by analytical geometry, and though quite complete, is somewhat difficult to follow with our present knowledge. Still it was the only proof available until the use of the *ratios* to represent the lines afforded the means of a much neater one, which, though not more complete, is much more easily understood.

Now let, as before, A O B be any angle, and A′ O A be any other angle. Then A′ O B will be the whole angle which is the sum of the other two.

Then—



M N drawn perpendicular to A O is the *sine* of angle β, N Q drawn from N perpendicular to O B is the *sine* of angle α, and M P drawn from M perpendicular to O B is the *sine* of angle $(\alpha + \beta)$. And these are represented by—

$$\frac{NQ}{ON} = \sin \alpha, \quad \frac{MN}{OM} = \sin \beta, \text{ and } \frac{MP}{OM} = (\sin \alpha + \beta).^*$$

---

[*] Notice that the *sines* of α and β can be made to meet at N, and those of β and $(\alpha + \beta)$ meet at M. This is a great convenience, and is gained by the substitution of the ratios for the lines. Compare the clearness and neatness of the figure now given with the complication and obscurity of the previous figure.

Also, $$\frac{\mathrm{O\,Q}}{\mathrm{O\,N}} = \textit{cosine of } a,$$

$$\frac{\mathrm{N\,O}}{\mathrm{O\,M}} = \textit{cosine of } \beta,$$

and $$\frac{\mathrm{O\,P}}{\mathrm{O\,M}} = \textit{cosine of } (a + \beta).$$

The line $\mathrm{M\,P} = \mathrm{M\,R} + \mathrm{R\,P}$, and $\mathrm{R\,P} = \mathrm{N\,Q}$;

$\therefore \mathrm{M\,P} = \mathrm{M\,R} + \mathrm{N\,Q}$. (1)

Also the angle at M = the angle a.

For $\angle \mathrm{R\,N\,O} = \angle \mathrm{N\,O\,Q}$; $\because$ R N parallel to O Q.

$\angle \mathrm{M\,N\,O} = 90°$.

Also $\angle \mathrm{M\,N\,R} + \angle \mathrm{N\,M\,R} = 90°$. $\because$ R is 90°.

$\therefore \angle \mathrm{M\,N\,O} = \angle \mathrm{M\,N\,R} + \angle \mathrm{N\,M\,R}$.

$\therefore \angle \mathrm{N\,M\,R} = 90° - \angle \mathrm{M\,N\,R} = \angle \mathrm{R\,N\,O} = a$.

$\therefore \angle \mathrm{N\,M\,R} = a$, and the *sin*, &c., of each are equal.

Therefore $$\frac{\mathrm{N\,R}}{\mathrm{M\,N}} = \sin \mathrm{N\,M\,R} = \sin a. \quad (2)$$

and $$\frac{\mathrm{M\,R}}{\mathrm{M\,N}} = \cos \mathrm{N\,M\,R} = \cos a. \quad (3)$$

These facts being premised, we come to the proof of our theorem.

To express sin, *&c., of* $(a + \beta)$ *in terms of the Ratios of* a *and of* β.

$$\sin (a + \beta) = \frac{\mathrm{M\,P}}{\mathrm{O\,M}} = \frac{\mathrm{M\,R} + \mathrm{N\,Q}}{\mathrm{O\,M}}. \quad \text{By (1)}$$

$$= \frac{\mathrm{M\,R}}{\mathrm{O\,M}} + \frac{\mathrm{N\,Q}}{\mathrm{O\,M}}.$$

But $\dfrac{\mathrm{M\,R}}{\mathrm{O\,M}}$ may be written $\dfrac{\mathrm{M\,R}}{\mathrm{M\,N}} \cdot \dfrac{\mathrm{M\,N}}{\mathrm{O\,M}}$, and $\frac{NQ}{OM}$ may be written $\frac{NQ}{ON}\cdot\frac{ON}{OM}$;

$$\therefore \sin(\alpha+\beta) = \frac{MP}{OM} = \frac{MR+NQ}{OM},$$

$$= \frac{NQ}{OM} + \frac{MR}{OM},$$

$$= \frac{NQ}{ON}\cdot\frac{ON}{OM} + \frac{MR}{MN}\cdot\frac{MN}{OM},$$

$$\left.\begin{array}{l} = \sin\alpha\,.\cos\beta + \cos\alpha\,.\sin\beta \\ = \sin\alpha\,.\cos\beta + \cos\alpha\,.\sin\beta \end{array}\right\} \text{By (2) \& (3).}$$

We may also express $\cos(\alpha+\beta)$ in this manner.

$$\text{For } \cos(\alpha+\beta) = \frac{OP}{OM} = \frac{OQ-PQ}{OM},$$

$$= \frac{OQ}{OM} - \frac{PQ}{OM} = \frac{OQ}{OM} - \frac{RN}{OM},$$

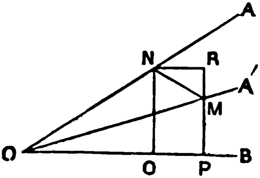
$$\because PQ = RN.$$

$$= \frac{OQ}{ON}\cdot\frac{ON}{OM} - \frac{RN}{MN}\cdot\frac{MN}{OM},$$

$$= \cos\alpha\,.\cos\beta - \sin\alpha\,.\sin\beta,$$

$$= \cos\alpha\,.\cos\beta - \sin\alpha\,.\sin\beta.$$

We know now how to express the sine and cosine of the *sum* of two angles, and in the same manner we can find expressions for the sin and cosine of the *difference* of two angles. But the figure we used for the *sum* is not the most convenient to explain the *difference*, which is best shown by the following :—



This is precisely the same figure as before, changing what requires to be changed. The smaller angle A′OB is the resulting angle now, and the line A′O may be supposed to have gradually approached the line AO, and after coinciding with it to have continued towards BO. During this change, R will gradually approach N, until, when A′O coincides with AO, RN = O. R will then become gradually greater, on the other side of NQ, as A′O approaches OB. MN also becomes first less and then greater. ONM still being the right angle, but on the other side of AO. Or we may imagine the former A′OA of the figure folded over on the line ON.

Then ∠ A′OB = what is left of ∠ AOB when AOA′ is taken from it; *i.e.*, angle α = AOB, angle β = A′OA, and A′OB = angle $(\alpha - \beta)$.

$$\sin\alpha = \frac{\mathrm{NQ}}{\mathrm{ON}},\ \sin\beta = \frac{\mathrm{MN}}{\mathrm{OM}},\ \sin(\alpha-\beta) = \frac{\mathrm{MP}}{\mathrm{OM}}.$$

$$\cos\alpha = \frac{\mathrm{OQ}}{\mathrm{ON}},\ \cos\beta = \frac{\mathrm{NO}}{\mathrm{OM}},\ \cos(\alpha-\beta) = \frac{\mathrm{OP}}{\mathrm{OM}}.$$

As before, the angle RMN = α, because QNR and MNO are each 90°, and, taking away MNQ,

$$\therefore \mathrm{ONQ} = \mathrm{RNM}.$$

But ONQ + QON = 90°, also RNM + RMN = 90°, and taking away ONQ and RNM, which are equal,

$$\therefore \mathrm{RMN} = \mathrm{NOQ} = \alpha.$$

$$\therefore \frac{\mathrm{NR}}{\mathrm{NM}} = \sin \mathrm{NMR} = \sin\alpha,$$

and

$$\frac{\mathrm{RM}}{\mathrm{NM}} = \cos \mathrm{NMR} = \cos\alpha.$$

Now we come to the proof of the formulæ for the *sine* and *cosine* of $(\alpha - \beta)$ in terms of *sin* and *cos* of α and β.

$$\begin{aligned} \sin(a-\beta) &= \frac{\mathrm{MP}}{\mathrm{OM}} = \frac{\mathrm{RP}-\mathrm{RM}}{\mathrm{OM}}, \\ &= \frac{\mathrm{NQ}-\mathrm{RM}}{\mathrm{OM}}, \qquad \because \mathrm{NQ} = \mathrm{RP}. \\ &= \frac{\mathrm{NQ}}{\mathrm{OM}} - \frac{\mathrm{RM}}{\mathrm{OM}}. \end{aligned}$$

But $\frac{\mathrm{NQ}}{\mathrm{OM}}$ may be written $\frac{\mathrm{NQ}}{\mathrm{ON}} \cdot \frac{\mathrm{ON}}{\mathrm{OM}}$,

and $\frac{\mathrm{RM}}{\mathrm{OM}}$ " " $\frac{\mathrm{RM}}{\mathrm{MN}} \cdot \frac{\mathrm{MN}}{\mathrm{OM}}$.

$$\begin{aligned} \sin(a-\beta) &= \frac{\mathrm{NQ}}{\mathrm{ON}} \cdot \frac{\mathrm{ON}}{\mathrm{OM}} - \frac{\mathrm{RM}}{\mathrm{MN}} \cdot \frac{\mathrm{MN}}{\mathrm{OM}}, \\ &= \sin a \,.\, \cos\beta - \cos a \,.\, \sin\beta. \end{aligned}$$

Also,
$$\begin{aligned} \cos(a-\beta) &= \frac{\mathrm{OP}}{\mathrm{OM}} = \frac{\mathrm{OQ}+\mathrm{QP}}{\mathrm{OM}}, \\ &= \frac{\mathrm{OQ}}{\mathrm{OM}} + \frac{\mathrm{QP}}{\mathrm{OM}}, \quad \text{But } \mathrm{QP} = \mathrm{NR}. \\ &= \frac{\mathrm{OQ}}{\mathrm{ON}} \cdot \frac{\mathrm{ON}}{\mathrm{OM}} + \frac{\mathrm{NR}}{\mathrm{MN}} \cdot \frac{\mathrm{MN}}{\mathrm{OM}}, \\ &= \cos a \,.\, \cos\beta + \sin a \,.\, \sin\beta. \end{aligned}$$

These four may be collected thus:—

$$\begin{aligned} \sin(a+\beta) &= \sin a \,.\, \cos\beta + \cos a \,.\, \sin\beta. \\ \sin(a-\beta) &= \sin a \,.\, \cos\beta - \cos a \,.\, \sin\beta. \\ \cos(a+\beta) &= \cos a \,.\, \cos\beta - \sin a \,.\, \sin\beta. \\ \cos(a-\beta) &= \cos a \,.\, \cos\beta + \sin a \,.\, \sin\beta. \end{aligned}$$

Notice, that the sign in the expression for the *sine* is + for $(a+\beta)$, and − for $(a-\beta)$; but is the reverse for the *cosine*, − for $(a+\beta)$ and + for $(a-\beta)$.

Examples.

(1.) Find the values for the sine and cosine of 120° in the terms of the ratios of 90° and 30°.

(2.) Find the above also in terms of 60° and 60°, and of 180° and 60°.

(3.) Find the ratios of 90° in terms of 45° and 45°.

(4.) Find the sine, cosine, and secant of 60°, in terms of 90° and 30°.

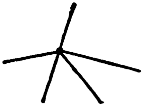
(5.) Express the sine, secant, and tangent of 270°, in terms of 180° and 90°.

(6.) Express the cosine and tangent of 180° in terms of 90° and 90°.

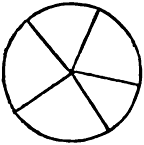
CHAPTER VIII.

MEASUREMENT BY DEGREES, GRADES, ETC.

IT is proved in Euclid, Book I., Prop. 13, that the two angles formed by any one straight line meeting another are together equal to two right angles. It was also shown (Euclid, Vol. I. page), that all the angles formed by any number of lines meeting at one point are, together, equal to four right angles.



Consequently, the angles formed at the centre of a circle, by any number of radii, are always equal to four right angles.



The total magnitude of these angles so formed at the centre of a circle, whatever be their number, is fixed and unalterable. But we may measure this fixed magnitude by any smaller magnitude of the same kind that we please.

The English divide this space into four quarters, each one being a right angle, and, further, divide each right angle, or *quadrant,* into ninety equal parts. To each of these parts is given the name of *degree,* expressed by °, so that 14° means 14 degrees, just as £14 means 14 sovereigns, or 14*d.* means 14 pence.

The French divide their right angle, or *quadrant,* into one hundred equal parts, and call each of these a *grade.*

The difference between the two systems may be shown thus :—

English.		*French.*	
1 *right angle*	= 90 *degrees*	1 *right ang.*	= 100 *grades*
1 *degree*	= 60 *minutes*	1 *grade*	= 100 *minutes*
1 *minute*	= 60 *seconds*	1 *minute*	= 100 *seconds*
Which may be written,		Which may be written,	
1 *right angle*	$= 90^\circ$	1 *right angle*	$= 100^g$
1°	$= 60'$	1^g	$= 100^{\backprime}$
$1'$	$= 60''$	$1^{\backprime}$	$= 100^{\backprime\backprime}$

The English method is convenient, and the French method inconvenient, to English mathematicians, because all their immense mass of astronomical and similar calculations are recorded in that manner; and in France, the French method is equally convenient, and the English method inconvenient, for a similar reason. The French method has, however, the advantage of being adapted to the decimal method of arithmetic, which the English is not.

An English *degree* is greater than a French *grade* in the ratio of 10 to 9; *i. e.,* 10 *grades* = 9 *degrees.*

An English *minute* is greater than a French *minute* in the ratio of 10,000 to 5400; *i.e.*, 10,000 French *minutes* = 5400 English *minutes*, or 100 French *minutes* = 54 English *minutes*.

An English *second* is greater than a French *minute* in the ratio of 1,000,000 to 324,000, or of 1000 to 324; *i.e.*, 1000 French *seconds* = 324 English *seconds*.

Therefore, any calculation made in one of these methods may be readily expressed in the other, since x *degrees* $= \frac{10}{9} x$ *grades*, and x *grades* $= \frac{9}{10} x$ *degrees*. If there be any odd minutes or seconds, the simplest method is to convert them into decimals of a degree or grade.

Examples.

(1.) Express 14·741 grades in degrees, minutes, &c.

(2.) Convert $14^\circ\ 14'\ 14''$ into grades, minutes, &c.

(3.) In a right-angled triangle, one of the acute angles is half of the other. Express the measurement of the three angles in the French and also in the English method.

(4.) Find the ratio between $67^\circ\ 14'$ and $77^g\ 14'$.

(5.) Reduce 101^g into degrees, &c.

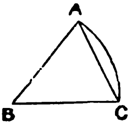
(6.) Reduce 84° into grades, &c.

CHAPTER IX.

CIRCULAR MEASUREMENT OF ANGLES.

If we have an angle, A B C, of such magnitude that the chord A C equals the radius A B or B C, we shall find the triangle A B C to be equilateral, and therefore 60° is the angle subtended by a chord equal to the radius.

But the *arc* A C is somewhat greater than the *chord* A C, and six such triangles as A B C will exactly fill the circle (of which the centre is B, and the radius A B), with the exception of the six spaces between the chords and the circumference.



Since each chord is equal to the radius, therefore the six chords are equal to thrice the diameter of the circle. But since each arc is greater than its chord, therefore the whole circumference is greater than thrice the diameter. Also, the angle subtended by an *arc* equal to the radius is less than 60°.

By actual calculation the ratio of the circumference to the diameter is found to be 3·14159+; *i. e.*, the circumference is 3·14159+ times the diameter.

Also the angle that subtends an arc equal in length to the radius is found to be 57·2957°+.

The first of these is an unalterable quantity; *i. e.*, in *every circle* the circumference is 3·14159 the diameter (or 6·28318+ times the radius), whatever be the system of calculation.

The second of these varies with the method of calculation; *i. e.*, if the circle be divided into 360°, then the angle spoken of is 57·2957+°; but if, as in the French method, the circle is divided into 400 grades, then the given angle is $\frac{10}{9} \times 57 \cdot 2957 +^{g}$ $= 63 \cdot 691 +^{g}$. But though the method of expression varies, the fact itself is unalterable; *i. e.*, the fact that the angle subtended by an arc equal to the radius is not quite, but very nearly, two-thirds of a right angle.

These two expressions, $57 \cdot 2957 +{}^\circ$ and $3 \cdot 14159 +$, occur very frequently in Trigonometrical calculations, and it is found convenient to express them by shorter symbols.

$57 \cdot 2957 +{}^\circ$ is expressed by ω, the long *o* of the Greek alphabet.

$3 \cdot 14159 +$ is expressed by π, the *p* of the Greek alphabet.

These two expressions are both constant, and have also a constant ratio to each other.

$$\begin{aligned} \text{For, arc of angle } \omega^\circ &= \text{radius}, \\ \text{also, circumference} &= \pi \,.\, \text{diameter}, \\ &= 2\,\pi \text{ radius}, \\ &= 2\,\pi \,.\, \text{arc of } \omega^\circ; \\ \therefore\ \omega^\circ \times 2\,\pi &= 360^\circ, \\ \textit{i.e.}, 57 \cdot 2957^\circ \times 3 \cdot 14159 &= 180^\circ. \end{aligned}$$

From this we get that—

$$\pi = \frac{180}{\omega},$$

$$\text{and } \omega^\circ = \frac{180^\circ}{\pi}.$$

Therefore, the circumference subtends an angle (360°) $2\,\pi$ times as great as $57 \cdot 2957 +{}^\circ$, and therefore the angle at the centre of a circle, *i.e.*, four right angles, is $2\,\pi$ times ω°.

$$\begin{aligned} \text{Therefore,}\quad \omega^\circ &= \frac{90^\circ}{\frac{1}{2}\,\pi} = \frac{\text{1 right angle}}{\text{half } 3 \cdot 14159 +}, \\ &= \frac{180^\circ}{\pi} = \frac{\text{2 right angles}}{3 \cdot 14159 +}, \\ &= \frac{360^\circ}{2\pi} = \frac{\text{4 right angles}}{2 \times 3 \cdot 14159 +}. \end{aligned}$$

We have from this another means of measuring angles, distinct from the measurement by degrees. We may say, now, of any angle, either that it is so many degrees, or such a fraction (proper or improper) of ω.

This ω may be considered as so many degrees, in which case the two methods of expression become the same, or it may be regarded as simply the particular angle subtended by an arc equal to the radius, in which case they are different.

Examples.

(1.) Express an angle of $36^\circ\ 10'$ as a fraction of ω°.

(2.) In a right-angled triangle, one of the acute angles is $\frac{3}{4}$ of the other. Express the respective magnitude of the three angles.

1. In the French method, grades, &c.
2. In the English method, degrees, &c.
3. As fractions of ω°.

(3.) Express $\dfrac{\omega^\circ}{4}$ in grades, and also in degrees.

(4.) In a circle of $8 \cdot 47$ radius, find the ratio of an arc of $4 \cdot 27$ to the circumference.

CHAPTER X.

ACTUAL VALUES OF RATIOS.

THE calculations necessary to find the actual values of sides and angles are frequently very long, and though not necessarily difficult, generally tedious.

One of the earliest methods of shortening these was to calculate the actual length of the sine and other lines belonging to angles between 0° and 90°.

Thus it was found that the sine of 30° is always $\frac{1}{2}$ or $\cdot 5$ of the radius, that the angle whose sine is $\frac{1}{4}$

or ·25 of the radius is between 14° and 15°, and that the sine which is ¾ or ·75 of the radius is between 48° and 49°. The exact length, to 7 places of decimals, of the sine and cosine for every degree and minute from 1′ to 90° was calculated, and the whole series collected into a table, thus:—

Degrees.	Length of Sine.	Degrees.	Length of Sine.
° ′		° ′	
0 0	·0000000	30 0	·5
0 1	·0002909		
0 2	·0005818	36 52	·5999549
0 3	·0008727	36 53	·6001876
0 4	·0011636		
5 44	·0998986	44 25	·6998711
5 45	·1001881	44 26	·7000789
11 32	·1999380	53 7	·7998593
11 33	·2002230	53 8	·8000338
17 27	·2998734		
17 28	·3001509	64 9	·8999386
		64 10	·9000654
23 34	·3998158		
23 35	·4000825	90 0	1

Here we have enough of the table extracted to show us that the sine increases about $\frac{1}{10}$ (one tenth) of the radius for every six degrees. But as the angle increases from 44° to 90° the sine increases much more slowly, taking 9° to increase from ·7 to ·8, 11° to increase from ·8 to ·9, and 26° to increase from ·9 to 1.

The values are expressed to 7 places of decimals, not because there are no more, but because, speaking generally, this is sufficiently accurate: also to avoid any difficulty arising from decimal calculations, both the sine and radius are multiplied by 10,000,000; *i. e.*, the radius is considered to be divided into ten million equal parts, and the given sine is expressed by saying how many of these parts it equals.

But this table will also give us, very readily, the value of the cosine of any angle. We have seen (page 42) that the cosine of any angle m° = the sine of $(90^\circ - m^\circ)$. Therefore we shall find that the same table which, *read downwards*, gives the lengths of the *sines*, will, *read upwards*, give the lengths of the *cosines*.

Degrees.	Sines.	Degrees.
° '		° '
5 44	·0998986	84 16
5 45	·1001881	84 15
11 32	·1999380	78 28
11 33	·2002230	78 27
17 27	·2998734	72 33
17 28	·3001509	72 32
23 34	·3998158	66 26
23 35	·4000825	66 25
30 0	·5	60
	Cosines.	

Thus we see that the sine of 30° and the cosine of

60° are alike ·5; the sine of 17° 27′ and the cosine of 72° 33′ are alike ·2998734, &c.

The tangents of all angles from 0° to 90° cannot be given in tables in this way, because they increase from 0 to ∞; *i.e.*, from nothing to infinity. But we do not so much need a table of tangents, since the equation—

$$\frac{sine}{cosine} = tangent,$$

will always enable us to find the tangent with ease and accuracy.

$$\text{Thus the tangent of } 0^\circ = \frac{sine}{cosine} = \frac{0}{1} = 0,$$

$$\text{the tangent of } 90^\circ = \frac{sine}{cosine} = \frac{1}{0} = \infty,$$

$$\text{the tangent of } 30^\circ = \frac{sine}{cosine} = \frac{5000000}{8660254} = \cdot 57 + ;$$

i.e., rather more than half the radius.

Also, since $cotangent = \frac{1}{tangent}$, $secant = \frac{1}{cosine}$, and $cosecant = \frac{1}{sine}$, we can very readily find the cotangent, secant, or cosecant of any angle.

Thus—

$$\text{cotan of } 60^\circ = \frac{1}{tan} = \frac{cos}{sin} = \frac{5000000}{8660254} = \cdot 57 +,$$

$$\text{secant of } 45^\circ = \frac{1}{cos} = \frac{1}{7071068} = \cdot 14 +,$$

and—

$$\text{cosec of } 40^\circ = \frac{1}{sine} = \frac{1}{6427876} = \cdot 15 + .$$

That we see that from the table of actual values of the sine from 0° to 90° we get, by inverting it, the values of the cosine, and also the values of the tangent, cotangent, secant, and cosecant.

But this calculation of the values of the sine, &c., did not aid very much in the actual working of trigonometrical calculations. The sines, &c., being expressed by seven figures, and the sides also requiring many figures to express their values correctly, the multiplication and division of these were still long and tedious operations, until the use of logarithms made them shorter and easier.

In Algebra it is proved that—

$$a^x \times a^y = a^{x+y},$$

and $$a^x \div a^y = a_{x-y}.$$

Upon these two theorems has been constructed a system of ready and accurate calculation of problems in multiplication and division, which is also applicable in extracting roots and finding powers of large numbers, and is therefore especially valuable in trigonometrical calculation. This system, called the system of Logarithms, is described in the next chapter.

Examples.

(1.) The length of the sine is ·3784. Give an approximate value for the angle.

(2.) In a right angle the perpendicular is $\frac{1}{5}$ of the hypothenuse. Find the approximate values of the two acute angles.

(3.) The base of a right-angled triangle is $\frac{1}{3}$ of the hypothenuse. Find approximate values for the angles.

(4.) One acute angle of a right-angled triangle is 23° 34′, and the hypothenuse is 49·4. Find the other angle, and approximate values of the other sides.

CHAPTER XI.

LOGARITHMS.

TRIGONOMETRY is built up on the twofold basis of Geometry and Algebra. Geometry is a pure science, the science of Space; Algebra is a pure science, the science of Number. Trigonometry is also a pure science, and may be called the science of Ratio. We saw, in talking of the Sixth Book of Euclid, that Ratio is more neatly and comprehensively expressed by Algebra than by Geometry. And so Trigonometry, which may be regarded as an extension of this Sixth Book, is compounded of the two sciences, each helping the other.

By a pure science is meant a science of *general truths* only. Thus, Algebra is a pure science; Arithmetic an applied science. $m + m = 2m$ is Algebra; $4 + 4 = 8$, or $3 + 3 = 6$, is Arithmetic. The one is a *general statement;* the other the *application* of the general statement to a particular case.

When we prove the general truths of Trigonometry, as the relations between the sides and angles, we are dealing with pure science; when we use the truths thus established for the purpose of finding the exact size of any particular angle, or the length of any given line, we use it as an applied science.

In this application we have very frequently to make very long and complicated calculations. In Algebra we find that an immense gain in time and accuracy is derived from the use, whenever possible, of the Binomial Theorem. Just in the same way, the use of Logarithms shortens very materially long calculations in Multiplication and Division.

The theory of Logarithms belongs to pure mathematics, and may be considered as a branch of advanced Algebra; but the subject requires to be discussed, at more or less length, in a treatise on Trigonometry, in order that we may thoroughly understand the method of use.

If we take the equation—

$$a^x = m,$$

we have a formula which expresses that some number a when raised to the xth power $= m$. We shall find that as a increases, m increases with it; also, as x increases, so m increases.

Also, if we know a and x, we know m; if we know a and m we can find x; generally, if we have any two of the three given, we can find the third.

The calculations of the successive values of x, in the different equations arising from $10^x = m$, as m gradually increases, are very long and intricate; but being once made, the values can be registered, and are thus known by simple reference.

The result, *practically*, is the same as if all possibly required sums in multiplication and division were worked out and the answers registered. The use of Logarithms is, however, a much more scientific system than any such registration of actual operations.

The practical utility of Logarithms consists in considering every *possible number* as a power of 10; that is, in the equation—

$$a^x = m;$$

putting $a = 10$, and then finding the value of x for every successive value of m. Thus—

$$10^x = 10.$$

Here $x = 1$; therefore, whenever m is more than 10, x is more than 1, and whenever m is less than 10, x is less than 1.

$$10^2 = 100.$$

Therefore, when m is greater than 100, x is greater than 2; and, if m is less than 100 but greater than 10, then x is less than 2 but greater than 1.

Also, $10^3 = 1000$, $10^4 = 10{,}000$, &c.: so that we know at once between what two whole numbers the value of x lies.

The values of x in the equation,

$$10^x = m,$$

being calculated for the successive whole numbers, beginning at 1, we get a table of Logarithms; thus,

if $10^x = 1$,
then $x = 0$;

if $10^x = 2$,
then $x = \cdot 3010300$;

if $10^x = 3$,
then $x = \cdot 4771213$;

if $10^x = 4$,
then $x = \cdot 6020600$;

if $10^x = 5$,
then $x = \cdot 6989700$.

Thus, the value of x is seen to gradually approach 1 until—

$$10^x = 10,$$

then $x = 1$.

In the logarithms from 0 to 10, the increase in the value of x is comparatively rapid; but in passing from 10 to 100, we shall find the increase much more gradual, since there will be ninety different values of x before we come to—

$$10^x = 100,$$

when $x = 2$.

Thus, $10^x = 11$;

$$\therefore x = 1 \cdot 0413927.$$

$$10^x = 12;$$
$$\therefore x = 1 \cdot 0791812.$$
$$10^x = 13;$$
$$\therefore x = 1 \cdot 1139434.$$
$$10^x = 54;$$
$$\therefore x = 1 \cdot 7323938.$$
$$10^x = 79;$$
$$\therefore x = 1 \cdot 8976271.$$
$$10^x = 99;$$
$$\therefore x = 1 \cdot 9956352.$$

From these we see, that the value of x when 10^x, equals any number between 10 and 100, passes gradually from 1 to 2, being the whole number 1 with a decimal, generally carried to the seventh place.

As 10^x increases from 100 to 1000, that is, as x increases from 2 to 3, we find the changes in the value of x much more slow, there being nine hundred values between 2 and 3.

Thus,

$$10^x = 101;$$
$$\therefore x = 2 \cdot 0043214.$$
$$10^x = 102;$$
$$\therefore x = 2 \cdot 0086002.$$
$$10^x = 103;$$
$$\therefore x = 2 \cdot 0128372.$$
$$10^x = 304;$$
$$\therefore x = 2 \cdot 4828736.$$
$$10^x = 739;$$
$$\therefore x = 2 \cdot 8686444.$$
$$10^x = 999;$$
$$\therefore x = 2 \cdot 9995655.$$

Between $10^x = 1000$ and $10^x = 10,000$, that is, between $x = 3$ and $x = 4$ there are *nine thousand* changes of value for x, beginning with—

$$10^x = 1001;$$
$$\therefore x = 3\cdot0004341,$$

and ending with—

$$10^x = 9999;$$
$$\therefore x = 3\cdot9999957,$$

and between $x = 4$ and $x = 5$ there are *ninety thousand* changes.

As to the practical advantage of using logarithms, it will be best shown by a few examples.

Example.—Find the product of 473 by 859.

Now, $10^x = 473,$

and $10^{x'} = 859;$

they, by algebra,

$$10^{x+x'} = 473 \times 859.$$

Therefore we find, by inspection, the logarithms of 473 and 859, and, by adding them together, find the logarithm of their product. Thus,—

$$\log 473 = 2\cdot6748611,$$
$$= 2\cdot9339932;$$

$$\therefore \log 473 + 859 = 5\cdot6088543.$$

And this number $5\cdot6088543$ is the logarithm or 406307. Therefore we say, that the product of 473 and 859 is 406307, and this we find without any labour of multiplication.

Again, required the product of 1741 and 8341.

$$\log 1741 = 3\cdot2407988,$$
$$\log 8341 = 3\cdot9212181;$$

$$\therefore \log 1741 \times 8341 = 7\cdot1620169,$$

and 7·1620169 is the log of 14521681;

$$\therefore 1741 \times 8341 = 14521681.$$

PART II.

INTRODUCTION.

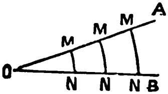
We have now acquired some familiarity with the technical terms of Trigonometry, and with the nature of the science. This will, it is hoped, enable us to commence a more scientific and concise study of the whole subject.

We are supposed to be familiar with the chief theorems of Geometry, to be able to solve readily both simple and quadratic Equations, and especially to have a thorough and familiar knowledge of what is meant by Ratio.

The chief aim in Part I. was to get a clear notion of the nature and use of Trigonometry. The object in Part II. is to prove, mathematically, the truth of its theorems, and to give a concise view of the elements of the science.

CHAPTER I.

MODES OF MEASURING ANGLES.



If any two lines A and B meet at O, they form at O, the point of meeting, an angle.

If with O as centre, and with any radius, O M, a circle be described, the portion of the circumference included between the two lines A O and B O varies with the magnitude of the angle.

If A O B be a right angle, M N is one quarter of the circumference.

If A O B be two right angles, M N is half the circumference.

If therefore we know the value of $\frac{\text{circumference}}{\text{M N}}$ we know the magnitude of the angle, and *vice versâ*.

The length of the radius O M is quite immaterial, since the greater the radius, the greater the circumference. Whatever number of circles be described with O as centre, the fraction $\frac{\text{circumference}}{\text{M N}}$ is the same for all.

O is the centre of the circle.
$\left.\begin{array}{l}\text{O M}\\ \text{O N}\end{array}\right\}$ are radii of the circle.
M N is an arc of the circle.

For convenience of practical expression, the circumference of every circle is divided into 360 equal portions, and each such portion is called a degree.

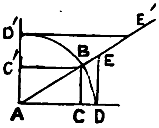
If $\text{M N} = \frac{\text{circumference}}{360}$, it is said to be an arc of one degree, which is expressed thus, 1°.

The angle M O N, which (when $\text{M N} = 1^\circ$) is $= \frac{\text{4 right angles}}{360}$, is said to be an angle of 1°.

Therefore $\frac{\text{4 right angles}}{360} = \frac{\text{circumference}}{360} = 1^\circ$, so that if we know the ratio of any angle to 4 right angles, we also know the ratio of the subtending arc to the circumference, and *vice versâ*. But the actual magnitude of an angle of m° is always the same, while that of an arc of m° varies with the radius.

Let B A C be any angle.

Describe, as before, with centre A and radius B,



the circle of which D′ B D is an arc; also,
draw B C perpendicular to A D,
„ B C′ „ „ A D′,
„ D E, the tangent to the circle at D,
„ D′ E′, „ „ „ D′, and
produce A B to meet D E in E and D′ E′ in E′.

Then—

$$\left.\begin{array}{lll} B\,D & \text{is the} & \textit{arc} \\ B\,C & ,, & \textit{sine} \\ D\,E & ,, & \textit{tangent} \\ A\,E & ,, & \textit{secant} \\ B\,C' = A\,C & ,, & \textit{cosine} \\ D'\,E' & ,, & \textit{cotangent} \\ A\,E' & ,, & \textit{cosecant} \end{array}\right\} \text{of the } \textit{angle } B\,A\,C = a.$$

Also—

$$\left.\begin{array}{lll} B\,A\,C & \text{is the} & \textit{angle} \\ B\,C & ,, & \textit{sine} \\ D\,E & ,, & \textit{tangent} \\ A\,E & ,, & \textit{secant} \\ B\,C' = A\,C & ,, & \textit{cosine} \\ D'\,E' & ,, & \textit{cotangent} \\ A\,E, & ,, & \textit{cosecant} \end{array}\right\} \text{of the } \textit{arc } B\,D.$$

Therefore we may speak of the *sine*, *secant*, and *tangent* of angle *a* as being the *sine*, *secant*, and *tangent* of the arc B D. In fact, we may, at any time, substitute angle *a* for arc B D, or arc B D for *a*.

But in every case the length of the radius A B is supposed to be known, and the *sine*, *tangent*, &c., are compared with it; so that, in fact, we have—

$$\left.\begin{array}{ll}
\frac{\mathrm{BC}}{\mathrm{AB}} & = \textit{sine} \\
\frac{\mathrm{DE}}{\mathrm{AB}} & = \textit{tangent} \\
\frac{\mathrm{AE}}{\mathrm{AB}} & = \textit{secant} \\
\frac{\mathrm{BC}'}{\mathrm{AB}} = \frac{\mathrm{AC}}{\mathrm{AB}} & = \textit{cosine} \\
\frac{\mathrm{D}'\mathrm{E}'}{\mathrm{AB}} & = \textit{cotangent} \\
\frac{\mathrm{AE}'}{\mathrm{AB}} & = \textit{cosecant}
\end{array}\right\} \begin{array}{c}\text{of angle } a, \\ \text{or} \\ \text{of arc B D.}\end{array}$$

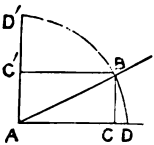
And by similar triangles, we have—

$$\left.\begin{array}{ll}
\frac{\mathrm{BC}}{\mathrm{AB}} & = \textit{sine} \\
\frac{\mathrm{BC}}{\mathrm{AC}} & = \textit{tangent} \\
\frac{\mathrm{AB}}{\mathrm{AC}} & = \textit{secant} \\
\frac{\mathrm{AC}}{\mathrm{AB}} & = \textit{cosine} \\
\frac{\mathrm{AC}}{\mathrm{BC}} & = \textit{cotangent} \\
\frac{\mathrm{AB}}{\mathrm{BC}} & = \textit{cosecant}
\end{array}\right\} \begin{array}{c}\text{of angle } a, \\ \text{or} \\ \text{of arc B D.}\end{array}$$

We spoke, in Part I. (page 52), of these latter ratios as *representing* the former. It was thought that this phraseology might be useful at that elementary stage; but it will now be seen that the two sets of ratios are identical, but that the latter is the more convenient mode of expression, inasmuch as it requires only the three lines A B, B C, and C A.

The angle $\beta = 90^\circ - a$, is called the *complement* of a. In same way a is the *complement* of β.

In all cases the *sine, tangent,* and *secant* of an angle are the *cosine, cotangent,* and *cosecant* of its complement, and *vice versâ.*



Theorem. $\quad sin^{2}$* $+ cos^{2} = 1.$

For

$$BC^2 + AC^2 = AB^2$$

$$\therefore \frac{BC^2 + AC^2}{AB^2} = 1;$$

$$\therefore \frac{BC^2}{AB^2} + \frac{AC^2}{AB^2} = 1;$$

$$\therefore sin^2 + cos^2 = 1.$$

In same way we have—

$$sec^2 - tan^2 = 1;$$

$$\therefore sec^2 = 1 + tan^2.$$

And also,

$$cosec^2 - cot^2 = 1;$$

$$\therefore cosec^2 = 1 + cot^2.$$

From these, as fundamental relations between the ratios, we are able to express any ratio in terms of any other. A complete table of the expressions is given in Part I. (page 45).

We have already seen that, for practical calculations, angles are measured by degrees, each degree

---

* By sin^2 is meant the square of the fraction representing the ratio.

$$=\frac{4\text{ right angles}}{360}.$$

Each degree is further divided into 60 equal parts, called minutes.

Each minute is again subdivided into 60 equal parts, called seconds.

The three marks, °, ′, and ″, are used to express these three divisions; thus, $40^\circ\ 14'\ 25''$.

We may also use, and frequently with advantage, a decimal mode of expression. Thus, we may write the above, $40^\circ\ 14{\cdot}416'+$, or as $40{\cdot}2402^\circ+$.

In French works on Trigonometry we find in use another division of the circumference, or of 4 right angles, which has the advantage of being at once expressed in decimals.

The 4 right angles are divided into 400 equal grades (each one being the 100th part of one right angle), each grade into 100 minutes, and each minute into 100 seconds. These are written $41^g\ 12^{`}\ 13^{``}$, which may be written as $41{\cdot}1213^g$, or as ·411213 of a right angle.

It has already been shown (page 59), that all that has been said is equally true, whatever be the magnitude of the angle (or the arc), whether greater or less than any number of right angles.

Also (page 64), not only can we express any one ratio in terms of any other, but we may express the ratios of one angle in terms of the ratios of others, as the ratios of 60° in terms of 40° and 20°, $90^\circ-30^\circ$, $30^\circ+30^\circ$, &c., &c.

Lastly, it has been seen (page 60), that simple cases of right-angled triangles are easily solved and calculated; but that more general problems required more familiarity with the inter-relations of the ratios.

The next chapter shows how to find some of these inter-relations, and how to express in the most convenient form.

CHAPTER II.

FACTS AND FORMULÆ.

WE have seen that it is possible to find the sides and angles of a triangle if we have given us three out of the six. We saw (page 21) that in simple cases this could be done by simple Geometry and Algebra; then (page 60), that more difficult cases could be solved, still without much trouble, by means of the simple ratios of Trigonometry, but also we have seen that any but the simpler cases require a more complete and familiar knowledge of the science.

In this chapter we will collect the more important of the Facts and Formulæ which we find useful in the solution of triangles. In this respect it will resemble a book of Euclid's Elements, being a collection of distinct theorems, each one being proved by means of some one or more of those that precede it.

To express the ratios of 2 A *in terms of* A.

$Sin\ 2A = 2\ sin\ A\ cos\ A.$

For $sin\ (A + B) = sin\ A\ cos\ B + cos\ A\ sin\ B$;
then, if $A = B$,

$$sin\ 2A = sin\ A\ cos\ A + cos\ A\ sin\ A,$$
$$= 2\ sin\ A\ cos\ A.$$

$Cos\ 2A = cos^2 A - sin^2 A.$

For $cos\ (A + B) = cos\ A\ cos\ B - sin\ A\ sin\ B$;
then, if $A = B$,

$$cos\ 2A = cos^2 A - sin^2 A.$$

$Cos\ 2A = 2\ cos^2 A - 1.$

For, $cos\ 2A = cos^2 A - sin^2 A$,
and $sin^2 = 1 - cos^2$;

$$\therefore \cos 2A = \cos^2 A - \sin^2 A,$$
$$= \cos^2 A - (1 - \cos^2 A),$$
$$= 2\cos^2 A - 1.$$

Cos $2A = 1 - 2\sin^2 A.$

For $\cos 2A = \cos^2 A - \sin^2 A,$
$$= 1 - \sin^2 A - \sin^2 A,$$
$$= 1 - 2\sin^2 A.$$

Therefore $\cos 2A = \cos^2 A - \sin^2 A,$
$$= 2\cos^2 A - 1,$$
$$= 1 - 2\sin^2 A.$$

$$\text{Tan } 2A = \frac{2\tan A}{1 - \tan^2 A}.$$

For $\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B},$

and, if $A = B$;

$$\therefore \tan 2A = \frac{\tan A + \tan A}{1 - \tan A \tan A},$$
$$= \frac{2\tan A}{1 - \tan^2 A}.$$

$$\text{Cosecant } 2A = \frac{1}{2\sin A \cos A}.$$

For $\sin 2A = 2\sin A \cos A;$

$$\therefore \text{cosec} = \frac{1}{2\sin A \cos A}.$$

$$\text{Sec } 2A = \frac{1}{\cos^2 A - \sin^2 A}.$$

For $\cos 2A = \cos^2 A - \sin^2 A;$

$$\therefore \sec 2A = \frac{1}{\cos^2 A - \sin^2 A},$$

$$= \frac{1}{2\cos^2 A - 1},$$

$$= \frac{1}{1 - 2\sin^2 A}.$$

$$\text{Cot } 2A = \frac{1 - \tan^2 A}{2\tan A}.$$

For $\tan 2A = \dfrac{2\tan A}{1 - \tan^2 A}$;

$$\therefore \cot 2A = \frac{1 - \tan^2 A}{2\tan A}.$$

By writing A for 2 A, and ½ A for A, we get, from the above,

$$\text{Sin } A = 2\sin \tfrac{1}{2}A \cos \tfrac{1}{2}A,$$

$$\cos A = \cos^2 \tfrac{1}{2}A - \sin^2 \tfrac{1}{2}A,$$

$$= 2\cos^2 \tfrac{1}{2}A - 1,$$

$$= 1 - 2\sin^2 \tfrac{1}{2}A,$$

$$\tan A = \frac{2\tan \frac{1}{2}A}{1 - \tan^2 \frac{1}{2}A},$$

$$\text{cosec } A = \frac{1}{2\sin \frac{1}{2}A \cos \frac{1}{2}A},$$

$$\sec A = \frac{1}{\cos^2 \frac{1}{2}A - \sin^2 \frac{1}{2}A},$$

$$\cot A = \frac{1 - \tan^2 \frac{1}{2}A}{2\tan \frac{1}{2}A}.$$

To express the ratios of A *in terms of* 2 A.

$$\text{Sin } A = \sqrt{\tfrac{1}{2}(1 - \cos 2A)}.$$

For
$$\begin{aligned} \sin 2\text{A} &= 2 \sin \text{A} \cos \text{A}, \\ &= 1 - 2 \sin^2\text{A}; \\ \therefore 2 \sin^2\text{A} &= 1 - \cos 2\text{A}; \\ \therefore \sin \text{A} &= \sqrt{\tfrac{1}{2}(1 - \cos 2\text{A})}. \end{aligned}$$

$$\textit{Cos}\ \text{A} = \sqrt{\tfrac{1}{2}(1 + \cos 2\text{A})}.$$

For
$$\begin{aligned} \cos 2\text{A} &= \cos^2\text{A} - \sin^2\text{A}, \\ &= 2 \cos^2\text{A} - 1; \\ \therefore 2 \cos^2\text{A} &= 1 + \cos 2\text{A}; \\ \therefore \cos \text{A} &= \sqrt{\tfrac{1}{2}(1 + \cos 2\text{A})}. \end{aligned}$$

$$\textit{Tan}\ \text{A} = \sqrt{\frac{1 - \cos 2\text{A}}{1 + \cos 2\text{A}}}.$$

For $\sin \text{A} = \sqrt{\tfrac{1}{2}(1 - \cos 2\text{A})}$,

and $\cos \text{A} = \sqrt{\tfrac{1}{2}(1 + \cos 2\text{A})}$,

also $\tan \text{A} = \dfrac{\sin \text{A}}{\cos \text{A}}$;

$$\begin{aligned} \therefore \tan \text{A} &= \frac{\sqrt{\tfrac{1}{2}(1 - \cos 2\text{A})}}{\sqrt{\tfrac{1}{2}(1 + \cos 2\text{A})}}, \\ &= \sqrt{\left(\frac{1 - \cos 2\text{A}}{1 + \cos 2\text{A}}\right)}. \end{aligned}$$

From the above, we have also,

$$\begin{aligned} \textit{Cosec}\ \text{A} &= \frac{1}{\sqrt{\tfrac{1}{2}(1 - \cos 2\text{A})}}, \\ \sec \text{A} &= \frac{1}{\sqrt{\tfrac{1}{2}(1 + \cos 2\text{A})}}, \\ \cot \text{A} &= \sqrt{\left(\frac{1 + \cos 2\text{A}}{1 - \cos 2\text{A}}\right)}. \end{aligned}$$

It will be noticed that all the ratios of A are expressed in terms of *cos* 2 A. It is, however, sometimes desirable to have an expression in terms of *sin* 2 A.

$$Sin\ A = \frac{\sqrt{(1 + sin\ 2\ A)} + \sqrt{(1 - sin\ 2\ A)}}{2}.$$

For—

$$(sin\ A \pm cos\ A)^2 = sin\ ^2A + cos\ ^2A \pm 2\ sin\ A\ cos\ A,$$
$$= 1 + sin\ 2\ A.$$

(since $(x + y)^2 = x^2 + y^2 + 2\,x\,y$,
$sin^2 + cos^2 = 1$,
and $sin\ 2\ A = 2\ sin\ A\ cos\ A$).

In same way $(sin\ A - cos\ A)^2 = 1 - sin\ 2\ A$;

$\therefore 2\ sin\ A = \sqrt{(1 + sin\ 2\ A)} + \sqrt{(1 - sin\ 2\ A)}$,
and $2\ cos\ A = \sqrt{(1 + sin\ 2\ A)} - \sqrt{(1 - sin\ 2\ A)}$.

From this, as above,

$$sin\ A = \frac{\sqrt{(1+sin\ 2\ A)} + \sqrt{(1-sin\ 2\ A)}}{2},$$

and $$cos\ A = \frac{\sqrt{(1+sin\ 2\ A)} - \sqrt{(1-sin\ 2\ A)}}{2}.$$

$$Tan\ A = \frac{\sqrt{(1+sin\ 2\ A)} + \sqrt{(1-sin\ 2\ A)}}{\sqrt{(1+sin\ 2\ A)} - \sqrt{(1-sin\ 2\ A)}}.$$

Also,

$$cosec\ A = \frac{2}{\sqrt{(1+sin\ 2\ A)} + \sqrt{(1-sin\ 2\ A)}},$$

$$sec\ A = \frac{2}{\sqrt{(1+sin\ 2\ A)} - \sqrt{(1-sin\ 2\ A)}}.$$

These are all given, in a tabular form, in Table 9, where will also be found a general collection of the equivalent values most useful in the Solution of Triangles, and in Trigonometry generally.

CHAPTER III.

SOLUTION OF RIGHT-ANGLED TRIANGLES.

BY the solution of a triangle is meant the process of finding the unknown sides or angles by means of relations known to exist between these and the sides or angles already known.

We have already seen that if we know any three of the six parts (three angles and three sides) of a triangle we can find the remaining three. The two exceptions to this are—

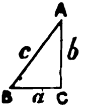
(1.) When the three angles only are given: in which case the number of triangles that will satisfy the given conditions is infinite.

(2.) When two sides and an angle opposite to one of them are given: in which case there are two triangles that will satisfy the given conditions.

The solution of right-angled triangles is, generally, easier than the solution of oblique-angled triangles, because we have two important theorems applicable to this case.

(1.) The square of the hypothenuse equals the sum of the squares of the two other sides.

(2.) The sum of the acute angles is 90°.



That is, $c^2 = a^2 + b^2$, and $A + B = 90°$.

Therefore we will first give the general formulæ for use in the various cases of a right-angled triangle.

(1.) To solve a right-angled triangle when the two sides are given (a, b, and C, given to find c, A, and B).

$$c = \sqrt{a^2 + b^2},$$

$$\frac{b}{a} = tan\ \mathrm{B},$$

$$\frac{b}{a} = cot\ \mathrm{A}.$$

If the values of a and b are small numbers, we get, very easily, the square root of the sum of their squares for the hypothenuse, and the quotient of b by a (or of a by b) will give the *tan* of one angle and the *cot* of the other.

But if these values, a and b, are large, this will require some tedious calculations in multiplication and division. These if done hastily are very liable to errors, and if done with care require time. Therefore it is better to have formulæ adapted to logarithmic calculation, which will enable us to do without the tedious calculation of long problems in multiplication and division, and which we may use to a greater or less degree of precision as we think fit.

Then we have—

$$tan\ \mathrm{B} = \frac{b}{a} \therefore \log tan\ \mathrm{B} = \log b - \log a;$$

$$\therefore \mathrm{L}\ tan\ \mathrm{B} = 10 + \log b - \log a.$$

Also, $\mathrm{A} = 90^\circ - \mathrm{B}$.

Lastly, $\frac{b}{c} = sin\ \mathrm{B} \therefore c = \frac{b}{sin\ \mathrm{B}}$

$$\therefore \log c = 10 + \log b - \mathrm{L}\ sin\ \mathrm{B}.$$

Example.—In a right-angled triangle, $a = 14$, $b = 68{\cdot}28$.

Required c, A, and B.

[illegible] formula $c = \sqrt{14^2 + 68{\cdot}28^2}$ would require the calculation of the squares of 14 and 68·28, and the extraction of the root of their sum.

The formula $\frac{b}{a} = \tan \text{B}$ would require the division of 68·28 by 14, and the finding the angle whose *tan* equals the quotient.

But, with the use of logarithms, the method of procedure is thus:—

$$\begin{aligned} \text{L} \tan \text{B} &= 10 + \log b - \log a, \\ &= 10 + 1{\cdot}8342935 - 1{\cdot}1461280, \\ &= 10{\cdot}6881655. \end{aligned}$$

Now this logarithm is between 10·6876734 (the L of *tan* 78° 24′), and 10·6883152 (L of *tan* 78° 25′). By calculating this difference, we find—

$$\text{L} \tan \text{B} = \text{L} \tan 78^\circ\, 24'\, 46'' + ;$$

$$\therefore \text{B} = 78^\circ\, 24'\, 46'' +.$$

$$\text{And A} = 90^\circ - \text{B} = 11^\circ\, 35'\, 14'' -.$$

For the third side c, we have the formula:—

$$\begin{aligned} \log c &= 10 + \log b - \text{L} \sin \text{B}, \\ &= 10 + 1{\cdot}8342935 - 9{\cdot}9910574, \\ &= 1{\cdot}8432361, \\ &= \log \text{ of } 69{\cdot}7; \end{aligned}$$

$$\therefore \log c = \log 69{\cdot}7;$$

$$\therefore c = 69{\cdot}7.$$

Therefore, in a triangle in which $a = 14$, $b = 68{\cdot}28$, $c = 90^\circ$, we have $c = 69{\cdot}7$, $\text{A} = 11^\circ\ 35'\ 14''$, $\text{B} = 78^\circ\ 24'\ 46''$.

(2.) To solve a right-angled triangle when the hypothenuse and one side are given (a, c, C, given to find b, A, and B).

Now, $$\begin{aligned} b &= \sqrt{c^2 - a^2}, \\ &= \sqrt{25{\cdot}23^2 - 8{\cdot}67^2}. \end{aligned}$$

Also, $\frac{a}{c} = \sin A = \cos B$.

Or, we may find A from $\sin A = \frac{a}{c}$, and then $B = 90° - A$, and $b = c \cos A$.

But if we wish to avail ourselves of logarithms, we must have formulæ adapted to logarithmic calculation; *i.e.*, formulæ composed of products or quotients, as logarithms are of no use in calculating sums and differences.

$$\sin A = \frac{a}{c};$$

$$\therefore \log \sin A = \log a - \log c;$$

$$\therefore L \sin A = 10 + \log a - \log c.$$

Also,
$$b = \sqrt{c^2 - a^2},$$
$$= \sqrt{(c-a)(c+a)};$$
$$\therefore \log b = \tfrac{1}{2} \log (c-a) + \tfrac{1}{2} \log (c+a).$$

Example.—Let $a = 8 \cdot 67$, $c = 25 \cdot 23$, $C = 90°$.

Now,
$$L \sin A = 10 + \log a - \log c,$$
$$= 10 + \cdot 9380191 - 1 \cdot 4019173,$$
$$= 9 \cdot 5361018,$$
$$= \sin 20° \, 5' \, 55'' +;$$
$$\therefore A = 20° \, 5' \, 55'' +.$$

Then,
$$B = 90° - A,$$
$$= 59° \, 54' \, 5''.$$

Finally,
$$\log b = \tfrac{1}{2} \log (c-a) + \tfrac{1}{2} \log (c+a),$$
$$= \frac{1 \cdot 2190603}{2} + \frac{1 \cdot 5301997}{2},$$
$$= 23 \cdot 69.$$

Therefore, if $a = 8 \cdot 67$, $c = 25 \cdot 23$, and $C = 90°$; also, $b = 23 \cdot 69$, $A = 20° \, 5' \, 55''$, and $B = 59° \, 54' \, 5''$.

(3.) To solve a right-angled triangle when a second angle and one of the sides be given (a, B, and C, given to find b, c, and A).

Now, $$\frac{b}{a} = \tan B;$$

$$\therefore b = a \tan B.$$

Also, $$\frac{c}{a} = \sec B;$$

$$\therefore c = a . \sec B.$$

And $$A = 90° - B.$$

This set of formulæ is adopted to logarithmic calculation, and we have—

$$\log b = \log a + \log tun\ B,$$
$$= \log a + L \tan B - 10;$$

also, $$\log c = \log a + \log \sec B,$$
$$= \log a + L \sec B - 10.$$

Example.—Let $a = 14{\cdot}7$, $B = 14° 27' 34''$, $C = 90°$

Then $$\log b = \log a + L \tan B - 10,$$
$$= 1{\cdot}1673173 + 9{\cdot}4113364 - 10.$$
$$= 5786537,$$
$$= \log 3{\cdot}79011;$$
$$\therefore b = 3{\cdot}79011.$$

Also, $$\log c = \log a + L \sec B - 10,$$
$$= 1{\cdot}1673173 + 10{\cdot}0139790 - 10,$$
$$= 1{\cdot}1812963,$$
$$= \log 15{\cdot}180,$$

and $$A = 90° - B = 75° 32' 26''.$$

Therefore, when $a = 14{\cdot}7$, $B = 14° 27' 34''$, and $C = 90°$; then $b = 3{\cdot}79011$, $c = 15{\cdot}18$, and $A = 75° 32' 26''$.

(4.) To solve a right-angled triangle when the hypothenuse and a second angle are given (c, A, and C, given to find a, b, and B).

Now, $$B = 90^\circ - A;$$

also, $$\frac{b}{c} = \sin B;$$

$$\therefore b = c \, . \sin B,$$

and $$\frac{a}{c} = \cos B;$$

$$\therefore a = c \, . \cos B.$$

These, adapted to logarithmic calculations, become—

$$\log b = \log c + \log \sin B,$$
$$= \log c + L \sin B - 10,$$

and $$\log a = \log c + L \cos B - 10.$$

Example.—Let $c = 34{\cdot}78$, $A = 14^\circ\, 14'\, 34''$, and $C = 90^\circ$.

Required a, b, and B.

Now, $B = 90^\circ - A = 75^\circ\, 45'\, 26''$;

also, $$\log b = \log c + L \sin B - 10,$$
$$= 1{\cdot}5413296 + 9{\cdot}9864412 - 10,$$
$$= 1{\cdot}5277708,$$
$$= \log 33{\cdot}7109.$$

Also $$\log a = \log c + L \cos B - 10,$$
$$= 1{\cdot}5413296 + 9{\cdot}3909641 - 10,$$
$$= 9322937,$$
$$= \log 8{\cdot}55645;$$
$$\therefore a = 8{\cdot}55645.$$

It must not be supposed that the method given here in any one of the four cases is the only, or even the best, method for all examples. Each triangle must be taken by itself, and practice will soon give sufficient familiarity with the facts and formulæ to enable us to see at once what is the most convenient method for any particular case.

Examples.

(1.) Given $a = 14{\cdot}71$, $b = 39{\cdot}8$
(2.) Given $a = 70{\cdot}09$, $c = 74{\cdot}103$
(3.) Given $b = 39{\cdot}47$, $c = 8{\cdot}4907$
(4.) Given $a = 7{\cdot}049$, $A = 79^\circ\ 14'\ 34''$
(5.) Given $a = 494{\cdot}72$, $B = 8^\circ\ 17'\ 9''$
(6.) Given $c = 784{\cdot}1$, $A = 79^\circ\ 14'\ 2''$
(7.) Given $c = 94{\cdot}73$, $B = 4^\circ\ 0'\ 17''$
(8.) Given $b = 7{\cdot}849$, $A = 79^\circ\ 0'\ 14''$
(9.) Given $b = 80{\cdot}047$, $B = 47^\circ\ 3'$
(10.) Given $a = 47{\cdot}47$, $b = 34{\cdot}894$
(11.) Given $b = 7{\cdot}1321$, $B = 7^\circ\ 4'\ 3''$
(12.) Given $c = 141{\cdot}3$, $B = 14^\circ\ 3'\ 49''$

Find the remaining angles and sides of these triangles, in all of which $C = 90^\circ$.

CHAPTER IV.

SOLUTION OF OBLIQUE-ANGLED TRIANGLES.

In a right-angled triangle the ratios of sides are identical with the ratios of the *sine*, *cosine*, &c.; but this is not the case with other triangles. Consequently the solution of these is more difficult and requires more familiarity with the facts and formulæ (explained in Chapter II.), and a readiness in applying those most suitable to each particular case.

(1.) Given the two sides and the included angle of any triangle to find the other two angles and third side. (b, c, A, given to find a, B, C)

Then $$\frac{\sin B}{\sin C} = \frac{b}{c};$$

$$\therefore \frac{\sin \mathrm{B} - \sin \mathrm{C}}{\sin \mathrm{B} + \sin \mathrm{C}} = \frac{b-c}{b+c};$$

$$\therefore \frac{\tan \frac{1}{2}(\mathrm{B}-\mathrm{C})}{\tan \frac{1}{2}(\mathrm{B}+\mathrm{C})} = \frac{b-c}{b+c}.$$

But—

$$\tan \tfrac{1}{2}(\mathrm{B}+\mathrm{C}) = \tan \tfrac{1}{2}(180^\circ - \mathrm{A}), \quad \because (\mathrm{B}+\mathrm{C} = 180^\circ - \mathrm{A})$$

$$= \cot \frac{\mathrm{A}}{2};$$

$$\therefore \tan \tfrac{1}{2}(\mathrm{B}-\mathrm{C}) = \frac{b-c}{b+c} \cot \frac{\mathrm{A}}{2};$$

$$\therefore \log \tan \tfrac{1}{2}(\mathrm{B}-\mathrm{C}) = \log(b-c) - \log(b+c) + \log \cot \frac{\mathrm{A}}{2};$$

$$\therefore \mathrm{L} \tan \tfrac{1}{2}(\mathrm{B}-\mathrm{C}) = \log(b-c) - \log(b+c) + \mathrm{L}^{*} \cot \frac{\mathrm{A}}{2}.$$

In this last equation all the terms in the right-hand side are known, and therefore we can find the value of $\frac{1}{2}$ (B−C). We can easily find $\frac{1}{2}$ (B+C) from B+C = 180°−A. A and B are then at once found.

This may seem complicated, but if *x*, *y*, *z*, &c., be substituted for the ratios it will be seen that the whole operation is simply an ordinary algebraic one; the only difficulty being the substitution of *tan* $\frac{1}{2}$ (B−C) for *sin* B − *sin* C, and of *tan* $\frac{1}{2}$ (B+C) for *sin* B+*sin* C. This is explained and proved on page 107.

The third side *a* is easily found from the equation $\frac{a}{c} = \frac{\sin \mathrm{A}}{\sin \mathrm{C}}$.

Example. — Given $b = 15{\cdot}81$, $c = 8{\cdot}47$, A = 14° 24′ 37″.

---

* Notice that we do not subtract 10 from L *cot*, because L *tan* appears on the other side of the equation, and the two 10's balance each other.

Required a, B, and C.

As before,
$$\frac{\sin \mathrm{B}}{\sin \mathrm{C}} = \frac{b}{c};$$

$$\therefore \frac{\sin \mathrm{B} - \sin \mathrm{C}}{\sin \mathrm{B} + \sin \mathrm{C}} = \frac{b-c}{b+c};$$

$$\therefore \frac{\tan \frac{1}{2}(\mathrm{B}-\mathrm{C})}{\tan \frac{1}{2}(\mathrm{B}+\mathrm{C})} = \frac{b-c}{b+c}.$$

But
$$\begin{aligned}\tan \tfrac{1}{2}(\mathrm{B}+\mathrm{C}) &= \tan \tfrac{1}{2}(180° - \mathrm{A}),\\ &= \cot \frac{\mathrm{A}}{2};\end{aligned}$$

$$\therefore \frac{\tan \frac{1}{2}(\mathrm{B}-\mathrm{C})}{\cot \frac{\mathrm{A}}{2}} = \frac{b-c}{b+c};$$

$$\begin{aligned}\therefore \tan \frac{(\mathrm{B}-\mathrm{C})}{2} &= \frac{b-c}{b+c} \cot \frac{\mathrm{A}}{2},\\ &= \frac{7 \cdot 34}{24 \cdot 28} \cot 7° \, 12' \, 18 \cdot 5''.\end{aligned}$$

As we saw before, this may be adapted to logarithmic calculation.

Then—

$$\begin{aligned}\mathrm{L} \tan \frac{\mathrm{B}-\mathrm{C}}{2} &= \log(b-c) - \log(b+c) + \mathrm{L} \cot 7° \, 12' \, 18'',\\ &= \log 7 \cdot 34 - \log 24 \cdot 28 + \mathrm{L} \cot 7° \, 12' \, 18'',\\ &= 8692317 - 1 \cdot 3852487 + 10 \cdot 891914,\\ &= \mathrm{L} \tan 67° \, 28' \, 19'';\end{aligned}$$

$$\therefore \frac{\mathrm{B}-\mathrm{C}}{2} = 67° \, 28' \, 19''.$$

Also,
$$\begin{aligned}\mathrm{B} + \mathrm{C} &= 180° - \mathrm{A},\\ &= 180° - 14° \, 24' \, 37'',\\ &= 165° \, 35' \, 23''.\end{aligned}$$

Therefore, $\mathrm{B} + \mathrm{C} = 165° \, 35' \, 23''$,

and $\mathrm{B} - \mathrm{C} = 134° \, 56' \, 38''$;

$$\therefore A = 14^\circ\,24'\,37'',$$
$$B = 150^\circ\,16'\,0{\cdot}5'',$$
and
$$C = 15^\circ\,19'\,22''$$
$$\overline{179^\circ\,59'\,59{\cdot}5''}.$$

The three angles amount to ·5″ less than 180°. This half a second is the sum of the decimal parts which we did not take into account, but which, of course, should be reckoned if minute accuracy is required.

Finally,
$$\frac{a}{c} = \frac{\sin A}{\sin C};$$
$$\therefore a = \frac{c \,.\, \sin A}{\sin C};$$
$$\therefore \log a = \log c + L \sin A - L \sin C,$$
$$= \log 7{\cdot}97642;$$
$$\therefore a = 7{\cdot}97642.$$

And the complete triangle is,

$$A = 14^\circ\,24'\,37'',$$
$$B = 150^\circ\,16'\,0{\cdot}5'',$$
$$C = 15^\circ\,19'\,22'';$$
$$a = 7{\cdot}97642,$$
$$b = 15{\cdot}81,$$
$$c = 8{\cdot}47.$$

(2.) Given two sides and the angle opposite to one of them to solve the triangle (a, b, A, given to find c, B, C).

The angle B is found from—

$$\frac{\sin B}{\sin A} = \frac{b}{a};$$
$$\therefore \sin B = \frac{b \,.\, \sin A}{a}.$$

Also, $C = 180° - (A + B)$,

and $c = \dfrac{a \,.\, sin\, C}{sin\, A}$ or $\dfrac{b \,.\, sin\, C}{sin\, B}$.

This case, however, requires special notice. See the remarks accompanying the example.

Example.—Given $A = 17° 14'$, $a = 8{\cdot}51$, $b = 27{\cdot}2$. Required B, C, and c.

As before,
$$\frac{sin\, B}{sin\, A} = \frac{b}{a},$$
$$= \frac{27{\cdot}2}{8{\cdot}51};$$
$$\therefore sin\, B = \frac{27{\cdot}2}{8{\cdot}51} sin\, A,$$
$$= \frac{27{\cdot}2}{8{\cdot}51} sin\, 17° 14'.$$
$$\therefore L\, sin\, B = \log 27{\cdot}2 - \log 8{\cdot}51 + L\, sin\, 17° 14',$$
$$= 9{\cdot}9763178,$$
$$= L\, sin\, 71° 14' 59'';$$
$$\therefore B = 71° 14' 59''.$$

Then, $C = 180° - (A + B)$,
$$= 180° - (17° 14' + 71° 14' 59''),$$
$$= 91° 31' 1'';$$

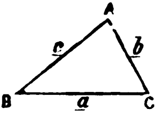
also, $c = \dfrac{a \,.\, sin\, C}{sin\, A}$;
$$\therefore \log c = \log a + L\, sin\, C - L\, sin\, A,$$
$$= \log 28{\cdot}7232;$$
$$\therefore c = 28{\cdot}7232$$

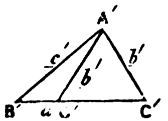
Thus the whole triangle is determined.

$A = 17° 14'$,
$B = 71° 14' 59''$,
$C = 91° 31' 1''$.

$$\begin{aligned} a &= 8\cdot51, \\ b &= 27\cdot2, \\ c &= 28\cdot7232. \end{aligned}$$

But there is another triangle besides this which will equally fulfil the conditions given. The reason of this (as given, so far as it concerns Geometry, in Euclid, Vol. I., page 11) is that the given angle is not included between the given sides, and therefore the side b may be either as in Fig. 1 or Fig. 2, the side a and angles A and C varying accordingly.





But our previous calculation should show this. If it does not, and we find that other triangles besides those found from given conditions will satisfy those conditions, it will suggest a doubt of the value of Trigonometry. But if we examine our former calculation we shall find that it will give us both the triangles.

Thus, $B = 17^\circ\, 14'$, $b = 8\cdot51$, $c = 27\cdot2$.

$$\text{Then, } \frac{\sin C}{\sin B} = \frac{c}{b}$$

$$= \frac{27\cdot2}{8\cdot51};$$

$$\therefore \sin C = \frac{27\cdot2}{8\cdot51} \sin 17^\circ\, 14';$$

$$\begin{aligned} \therefore L \sin C &= \log 27\cdot2 - \log 8\cdot51 + L \sin 17^\circ\, 14', \\ &= 9\cdot9763178, \\ &= L \sin 71^\circ\, 14'\, 59''. \end{aligned}$$

But 9·9763178 also equal the L of an angle as much greater than 90° as 71° 14′ 59″ is less; *i.e.*, $\sin 90^\circ - x = \sin 90^\circ + x$. But, also, if a be greater than b, B cannot be greater than 90°, therefore we can only take the smaller value.

$$\therefore \sin 71^\circ\,14'\,59'' = \sin 108^\circ\,45'\,1'',$$
$$(\text{for } 90^\circ - 18^\circ\,45'\,1'' = 71^\circ\,14'\,59'',$$
$$\text{and } 90^\circ + 18^\circ\,45'\,1'' = 108^\circ\,45'\,1'');$$
$$\therefore C = 90^\circ \pm 18^\circ\,45'\,1'',$$
$$= 71^\circ\,14'\,59'' \text{ or } 108^\circ\,45'\,1''.$$

Therefore, if we know only two sides and one angle of a triangle, we may not completely know the triangle, unless the angle be the one included by the two sides.

But though we cannot find the triangle itself, we know that it must be one of two, both of which we can find.

We have now two triangles.

In (1) $B = 17^\circ\,14'$, $C = 71^\circ\,14'\,59''$, $b = 8{\cdot}51$, $c = 27{\cdot}2$.
In (2) $B = 17^\circ\,14'$, $C = 108^\circ\,45'\,1''$, $b = 8{\cdot}51$, $c = 27{\cdot}2$.

Whichever value we take for C, we can easily find A and a.

For $$A = 180^\circ - (C + B),$$

and $$a = \frac{b \,.\, \sin A}{\sin B}.$$

(3.) Given two angles and a side to find the remaining sides and angle (A, B, c, given to find C, a, b).

In this case the given side is the one adjacent to both angles.

Then, $$C = 180^\circ - (A + B),$$ $$a = \frac{c \,.\, sin\ A}{sin\ C},$$ $$b = \frac{a \,.\, sin\ B}{sin\ A}.$$

Example.—Given $A = 64^\circ\ 17'$, $B = 81^\circ\ 31'\ 14''$, $c = 17{\cdot}43$.

Required C, a, and b.

Then, $$\begin{aligned} C &= 180^\circ - (A + B), \\ &= 180^\circ - 145^\circ\ 48'\ 14'' \\ &= 34^\circ\ 11'\ 46''; \end{aligned}$$

also, $$\begin{aligned} a &= \frac{c \,.\, sin\ A}{sin\ C}, \\ &= \frac{17{\cdot}43 \times sin\ 64^\circ\ 17'}{sin\ 34^\circ\ 11'\ 46''}; \end{aligned}$$

$$\therefore \log a = \log 17{\cdot}43 + L\ sin\ 64^\circ\ 17' - L\ sin\ 34^\circ\ 11'\ 46'',$$ $$= \log 27{\cdot}9409;$$ $$\therefore a = 27{\cdot}9409.$$

Finally, $$\begin{aligned} b &= \frac{c \,.\, sin\ B}{sin\ C}, \\ &= \frac{17{\cdot}43 \times sin\ 81^\circ\ 31'\ 14''}{sin\ 34^\circ\ 11'\ 46''}; \end{aligned}$$

$$\therefore \log b = \log c + L\ sin\ 81^\circ\ 31'\ 14'' - L\ sin\ 34^\circ\ 11'\ 46'',$$ $$= \log 29{\cdot}9755,$$ $$\therefore b = 29{\cdot}9755.$$

Therefore, $$\begin{aligned} A &= 64^\circ\ 17', \\ B &= 81^\circ\ 31'\ 14'', \\ C &= 34^\circ\ 11'\ 46'', \\ a &= 27{\cdot}9409, \\ b &= 29{\cdot}9755, \\ c &= 17{\cdot}43. \end{aligned}$$

(4.) Given two angles and a side opposite to one of them to find the remaining sides and angle (A, B, a, given to find C, b, c).

Then, $$C = 180° - (A + B),$$
$$b = \frac{a \cdot \sin B}{\sin A},$$
$$c = \frac{a \cdot \sin C}{\sin A}.$$

Example.—Given $A = 37° 14'$, $B = 51° 17'$, $a = 49{\cdot}7$. Required C, b, and c.

Then, $$C = 180° - (A + B),$$
$$= 180° - 88° 31',$$
$$= 91° 29'.$$

Also, $$b = \frac{a \cdot \sin B}{\sin A},$$
$$= \frac{49{\cdot}7 \times \sin 51° 17'}{\sin 37° 14'};$$

$$\therefore \log b = \log 49{\cdot}7 + L \sin 51° 17' - L \sin 37° 14',$$
$$= \log 64{\cdot}0898;$$
$$\therefore b = 64{\cdot}0898.$$

And $$c = \frac{a \cdot \sin C}{\sin A},$$
$$= \frac{49{\cdot}7 \times \sin 91° 29'}{\sin 37° 14'};$$

$$\therefore \log c = \log 49{\cdot}7 + L \sin 91° 29' - L \sin 37° 14';$$
$$= \log 78{\cdot}417;$$
$$\therefore c = 78{\cdot}417.$$

Therefore, $$A = 37° 14',$$
$$B = 51° 17',$$
$$C = 91° 29',$$

$$a = 49 \cdot 7,$$
$$b = 64 \cdot 0898,$$
$$c = 78 \cdot 417.$$

It will be necessary to recur again to the particular case of one angle being greater than 90°.

CHAPTER V.

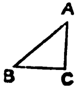
PRACTICAL APPLICATION OF TRIGONOMETRY.

We come now to the practical use of Trigonometry, to the discussion of the methods by which we are enabled to measure angles and distances that are inaccessible by ordinary methods, such as measuring the height of a tower or wall that we cannot approach, or the width of a river that we cannot cross.

The great principle is to fix upon three points, and to regard them as the three angular points of a triangle. Any general remarks will, however, be better understood after we have discussed a few examples.

Example.—Required to find the width of a river without crossing it.

Select some object (A) on the opposite bank, and observe it from two points (B and C) on this side, noting the magnitude of the two angles B and C, and their distance from each other. The breadth of the river will then form one side of the triangle thus formed, if B or C be opposite A.



Then, in the triangle A B C, we know two angles and the included side, and by the ordinary method we can find the other sides and the remaining angle.

Thus, let $C = 90°$, $B = 47° 4'$, and $a = 89 \cdot 7$.

We do not, however, want any but the side b, since our object is not to solve the triangle A B C, but to find the breadth of the river; that is, the side b.

$$\text{Then,} \quad \frac{b}{a} = \frac{\sin B}{\sin A};$$

$$\therefore b = \frac{a \,.\, \sin B}{\sin A};$$

$$\begin{aligned} \therefore \log b &= \log a + L \sin B - L \sin A, \\ &= \log 89 \cdot 7 + L \sin 47° 4' - L \sin 42° 56' \\ &= \log 96 \cdot 4161; \\ \therefore b &= 96 \cdot 4161; \end{aligned}$$

which is the width of the river, as required.

But the angle C may not be 90°; that is, the triangle may not be a right-angled one, for the point A may not be opposite either B or C, or the river may wind so that B C may not be perpendicular to A C or to A B.

Suppose $a = 89 \cdot 7$, $B = 49° 6'$, and $C = 54° 17'$.

$$\begin{aligned} \text{Then,} \quad A &= 180° - (B + C), \\ &= 76° 37', \end{aligned}$$

$$\text{and} \quad \frac{b}{a} = \frac{\sin B}{\sin A};$$

$$\therefore b = \frac{a \,.\, \sin B}{\sin A};$$

$$\begin{aligned} \therefore \log b &= \log a + L \sin B - L \sin A, \\ &= \log 89 \cdot 7 + L \sin 49° 6' - L \sin 76° 37', \\ &= \log 62 \cdot 6926; \\ \therefore b &= 62 \cdot 6926; \end{aligned}$$

which is the width of the river in the slanting direction A C. This supposes that it is either impracticable or undesirable to measure it at right angles to its course.

But the distance directly across the river from A is easily found, since the line of this distance forms with the line b and part of a a smaller triangle, of which B′ will be 90°, and $A' = 90° - c$. In this smaller triangle we know one side b and the two adjacent angles A′ and C. The side c' is the direct distance of the river.

Thus, $b = 62 \cdot 6926$, $C = 54° \ 17'$, $A' = 45° \ 43'$.

(By A′ and B′ are meant the angles in the small triangle.)

$$\text{Then,} \quad B = 180° - (A + C),$$
$$= 90°;$$
$$\text{also,} \quad \frac{c'}{b} = \frac{\sin C}{\sin B};$$
$$\therefore c' = \frac{b \cdot \sin C}{\sin B};$$
$$\therefore \log c' = \log 62 \cdot 6926 + L \sin 54° \ 17' - L \sin 90°,$$
$$= \log 50 \cdot 8997;$$
$$\therefore c' = 50 \cdot 8997;$$

which is therefore the width of the river measured directly across.

Example.—It is desired to ascertain the height of a tower, the top of which is inaccessible.

Take any convenient point (C) at the foot of the tower, and any other (B) at a convenient distance from it. Measure the distance between these points (C and B).

In the triangle A B C, the side b is the height of the tower, and we know the side a, the angle B, and the angle C. B is found by observing the top of the tower from that point, and measuring the angle which the line of vision makes with the ground.

C is a right angle, being the angle of the wall with the ground.

Suppose $a = 97 \cdot 4$, $B = 43° 14'$, and $C = 90°$.

$$\text{Then, } \frac{b}{a} = \frac{\sin B}{\sin A};$$

$$\therefore b = \frac{a \cdot \sin B}{\sin A};$$

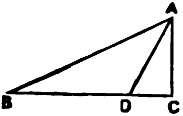
$$\begin{aligned} \therefore \log b &= \log a + L \sin B - L \sin A, \\ &= \log 97 \cdot 4 + L \sin 43° 14' - L \sin 46° 46' \\ &= \log 81 \cdot 6196; \\ \therefore b &= 81 \cdot 6196; \end{aligned}$$

which is the height of the tower.

It may be that the highest point of the tower, of which the height is desired, may not be directly above the outer wall, that is, the angle C may not be a right angle. In this case we must use a second triangle, as in the last example. Of this second triangle, the side *b*, the height of the tower, and the distance of the tower from the outer wall, form the three sides, of which only *b* is known. But we know all the angles, since one is 180°−C, the second 90°, and the third 90°−C. From these we can easily find the side b', which is the perpendicular height of the tower.

The two cases of this example are precisely identical with the two cases of the last proposition, the only difference being that in the case of the river the triangle is measured horizontally, while in the case of the tower it is perpendicular. In the one case, A, B, and C, are all on the ground; in the other only B and C are on the ground, A being vertically above it.

But it may happen that the top of the tower is not observable from the base of the outer wall; most probably it would not be. In this case we must take two points away from the wall, thus:—

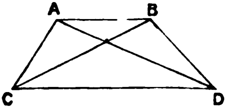


Observe the angle A B C, also the angle A D C, then also the angle A D B is known. Measure the base B D; then we have in the triangle A B D two angles and the adjacent side; this will give us A D. Then in the triangle A D C we know A D C and D A C; also C is 90°. Also we know A D, and so can easily find A C, the required height.

But it may also happen that the two points of observation B and D cannot be taken in the same line with C. In this case they may be taken at any convenient points. The course of procedure is, as before, to find A B in the triangle A B D, and then consider it as a side of the triangle A B C, and so to find A C, as required.

Example.—To find the distance between two points that are both visible, but also both inaccessible to the observer. Let A and B be the two points.

Take two convenient points from each of which A and B are both visible. Let these points of observation be C and D.



At C measure the angles A C B and D C B. At D measure the angles A D C and B D A. Also measure C D. From these data, the object is to find the length of A B.

Now, in the triangle A C D we know the side C D, and the angles A C D and A D C, therefore we can find A C.

Then, in the triangle B C D we know the side C D, and the angles B C D and B D C, therefore we can find B C.

Lastly, in the triangle B A C we know A C and B C, and the angle B C A, therefore the side A B can be found.

But A B is the distance between the two points A and B, which was required to be known.

PART III.

CHAPTER I.

SPHERICAL GEOMETRY.

IF a circle be allowed to revolve round one of its own diameters, the figure so described will be a sphere, a figure defined by saying that every point in its surface is equally distant from the centre of the generating circle, and of which we have a familiar example in a ball.

If a sphere be cut by a plane, the section is a circle; if the plane pass through the centre of the sphere, the circle of the section is greater than if it does not.

All circles that pass through the centre of the sphere are equal in size, and are called great circles.

When a sphere is cut by a plane, the centre of the circle made by the section is found by drawing a perpendicular from the centre of the sphere to the plane of the circle.

Circles are said to be nearer to the centre of the sphere (and are greater) as this perpendicular distance decreases, and, *vice versâ,* the circles decrease in size as their distance from the centre increases.

A great circle (*i.e.,* one passing through the centre) divides a sphere into two equal parts, called *hemispheres.*

Two great circles at right angles to each other divide the sphere into four equal parts, called *quadrants.*

Two great circles, whose inclination to each other is less than a right angle, enclose a portion of the sphere which is less than a quadrant, and which is called a *lune.*

The surface of a sphere is equal to the product of the circumference and diameter of the generating circle, or (which is the same thing) of a great circle.

Let d be the diameter, and c the circumference, then the surface of the sphere $= d\,c = 3{\cdot}14\,d^2$.

Example.—Let the diameter of a circle be 7 inches,
then the circumference $= 7 \times 3{\cdot}14 = 21{\cdot}98$,
and the surface of the sphere $= 21{\cdot}98 \times 7 = 153{\cdot}86$.

The surface of a sphere is equal to four times the area of a great circle ; i.e., *of the generating circle.*

Let d be the radius, then $d \times {\cdot}7854 =$ area of circle, and surface of sphere $= d \times {\cdot}7854 \times 4$.

Example.—Let the diameter of a circle $= 7$,
the circumference $= 21{\cdot}98$, (as above)
the area of the circle $= 7^2 \times {\cdot}7854 = 38{\cdot}4846$,
and surface of sphere $= 38{\cdot}4846 \times 4$,
$= 153{\cdot}9384$.

The difference between $153{\cdot}9384$ and $153{\cdot}86$ as given above, is owing to the decimals not being taken to a sufficient number of places; but, also, in this kind of calculation, we seldom get complete equality or complete values, partly because of the length of the decimals, and partly because they are frequently recurring decimals, and therefore the values are all more or less approximations.

The solid content of a sphere is equal tc

$$\frac{\textit{surface} \times \textit{radius}}{3}.$$

Let d be the diameter of a great circle, and c the circumference,

then the surface $= c\,d$,

the radius $= \frac{d}{2}$,

and the solidity of the sphere $= \frac{c\,d \times d}{3 \times 2}$,

$$= \frac{c\,d^2}{6}.$$

Example.—Let the diameter be 7,

then the radius $= \frac{7}{2}$,

the surface $= 153{\cdot}9$,

and the solidity $= \frac{153{\cdot}9 \times 7}{3 \times 2}$,

$$= 179{\cdot}5.$$

Surfaces of spheres are to one another as the squares of their radii, and their solid contents as the cubes of their radii.

If S and S′ be the superficies (or surfaces) of two spheres, and R . R′ their respective radii, then—

$$S : S' :: R^2 : R'^2.$$

Also, if C and C′ be their respective solidities, then—

$$C : C' :: R^3 : R'^3.$$

Example.—Let there be two spheres, one of 5 inches radius, and the other of 7 inches; then the surfaces are as 25 to 49, and the solid contents as 125 to 343.

From this we see that if we double the radius of a sphere we increase the surface fourfold, and the solid contents eightfold.

Also to halve the radius is to quarter the surface and to reduce the solidity to one-eighth.

As this theorem is very important, and not always very readily comprehended in all its fulness, we give here a few more examples.

Table of Surfaces and Solidities of Spheres.

Radii.	Surfaces.	Solidities.
feet.	square feet.	cubic feet.
1	12·56	4·19
2	50·24	33·49
3	113·04	113·04
4	200·96	267·88
5	314·00	523·33
6	452·16	904·32
7	615·24	1435·56
8	803·84	2143·57
9	1017·36	3052·08
10	1256·00	4186·66

SPHERICAL TRIANGLES.

If two great circles cross each other they enclose, as we have seen, a portion of the sphere, called a lune, resembling in shape a wedge, the width of the thicker edge of which varies from 0 to half the circumference; never being quite so small as 0, never being quite so large as half the circumference.

If a third circle cross two intersecting circles, the portion cut off is a triangular wedge, differing from a lunar wedge in that the outer surface is a triangle, while the inner edge becomes a point.

The three sides of the triangle are arcs of circles, subtended at the centre by angles. The angle subtending each arc is the same as the angle of the planes of the other two circles.

Therefore, in a spherical triangle, there are six angles to be considered:

(1). The three angles of the triangle.

(2). The three angles corresponding with the three sides.

We have seen (page 90), that we consider the sine of any angle as also the sine of the arc subtending the angle.

Therefore, we have, in every spherical triangle, six sines:

(1). Three of the sides.

(2). Three of the angles.

Exercises.

(1). Find the superficial extent of a sphere of which the radius is 4·7 inches.

(2). Find the solidity of a sphere having a radius of 81·73 yards.

(3). If a cubic foot weigh 10 lbs. find the weight of a sphere the diameter of which is 819 feet.

(4). How many square yards in the superficies of a sphere of 5·8 feet radius?

(5). One globe has a radius of 2·8 inches, and another of 4·1 inches. How much is one superficies greater than the other? and by how much does the solidity of one exceed that of the other?

CHAPTER II.

SPHERICAL TRIANGLES.

A SPHERICAL triangle resembles a plane triangle in having three sides and three angles, but it differs from it in that its sides are arcs of circles.

This difference is a very important one, since we have seen (page 91) that, for any angle, we may consider the sine, cosine, tangent, &c., as belonging either to the angle or to the arc of the circumference corresponding to it; also, that every arc may be considered as having a corresponding angle.

Therefore, in a spherical triangle, not only the angles, but also the sides, have ratios; *i.e.*, we have sines, cosines, tangents, &c., for the three angles, A, B, C, and also sines, cosines, tangents, &c., for the three sides, *a*, *b*, *c*.

But the sines, &c., of the arcs of circles are the same as those of the corresponding angles at the centre. Therefore, we may consider the six parts of a spherical triangle as six angles.

Thus, A, B, C, the three angles of the triangle,
and * A′, B′, C′, the three angles corresponding to the sides.

But this view may be still further extended, with a very important result.

The three angles, A′, B′, C′, corresponding to the arcs *a*, *b*, *c*, are all supposed to be at the centre of the sphere. This follows from the fact that we consider here only triangles formed by *great circles*. If a triangle be formed (as any number of triangles may be) by three circles, of which one, two, or all, do not pass through the centre, such a triangle is not considered as a spherical triangle, though it be formed on the surface of a sphere by three intersecting circles. The technical term "spherical triangle" is confined to triangles formed by three intersecting great circles.

Therefore, in every spherical triangle, we have,
A, B, C, the three angles of the triangle,
and A′, B′, C′, the three angles forming a solid angle

---

* We use A′, B′, C′, as the angles corresponding to the sides *a*, *b*, *c*. Any other letters, as D, E, F; M, N, O, &c., would do; but A′, B′, C′, are more symmetrical.

at the centre of the sphere, and corresponding to the sides a, b, c, respectively.

This brings us to the consideration of a solid angle at the centre of the sphere. Just as we connect a plane angle with an arc of a circle, so we shall now be able to connect a solid angle and a spherical triangle.

If we know the radius of a circle and the length of any arc, we know the angle at the centre corresponding to that arc, and *vice versâ*. Just so, if we know the radius of a sphere, the data of a solid angle at the centre will give us the sides and angles of the corresponding spherical triangle, and the sides and angles of a spherical triangle will give us the data of the corresponding solid angle.

Every spherical triangle is made by three arcs of great circles. Each of these arcs has a corresponding angle at the centre of its circle, but the centre of a sphere is the common centre of all its great circles, so that these three angles form a solid angle at the centre of the sphere. The three sides may be expressed by a, b, c, and the corresponding angles by A′, B′, C′, as before.

This gives us one correspondence between a spherical triangle and its solid angle.

But each angle is in the same plane as its corresponding arc; therefore the three angles of the solid angle are at the same inclination to each other as the sides of the triangle; that is, the inclination of the three faces of the solid angle to each other are the same as the three angles of the triangle.

This gives us another correspondence between the spherical triangle and its solid angle.

Therefore, in every spherical triangle, the sides a, b, c, correspond to the three angular faces of a solid angle, and the three angles A, B, C, are the same as the angles of inclination of these three faces. So that as we speak of an angle or of its arc, so we may speak of a spherical triangle or of its solid angle, and *vice versâ*.

In every spherical triangle, each part, A, B, C, *a*, *b*, *c*, has the six ratios, sine, cosine, tangent, secant, cosecant, and cotangent.

In speaking of the sine, cosine, &c., of a side, we consider the side as an arc of a circle.

All the properties we have proved for the ratios of plane angles are equally true of the ratios belonging to the angles of spherical triangles and to the solid angles belonging to them; but we must extend our notions of triangles somewhat further than has been hitherto required for plane triangles.

We may get a practical idea of a spherical triangle, and its corresponding solid angle, by taking a Dutch cheese and cutting it half-way through in three different directions, so that each of the three planes of section crosses the other two.

In this way a piece will be quite cut away from the cheese; a piece in the shape of a triangular pointed wedge. If this piece be removed, it will leave a triangular hole bounded at the surface by three curved lines, and having three flat sides terminating together in a point at the centre of the cheese. The three curved lines will represent the spherical triangle, the point at the centre will represent the corresponding solid angle.

The piece removed will terminate in a three-faced solid angle, and the opposite end will be a curved surface bounded by three curved lines forming the corresponding spherical triangle.

In the next chapter will be given a set of formulæ for the solution of spherical triangles.

It will be found somewhat different from the formulæ and method of solution in the case of plane triangles; but each method will be found to illustrate the other, and assist the comprehension of it.

CHAPTER III.

RATIOS OF SPHERICAL TRIANGLES.

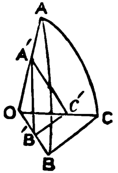
In the preceding chapter we explained how spherical triangles were measured, and their sides and angles represented by ratios in very much the same manner as plane triangles.

We have now, in this chapter, to explain three things:

1st. How to find the ratios.
2nd. How to establish some fundamental theorems.
3rd. How to apply these theorems to the solution of triangles.

I. *How to find the Ratios of a Spherical Triangle.*

For the sake of simplicity, we will take, first, the case of right-angled triangles; *i.e.*, spherical triangles in which one of the angles is a right angle.



Let A B C be a triangular portion of the surface of a sphere, and let O be the centre of the sphere.

Then O A, O B, O C, are radii of the sphere, and are all of equal length; and if we take a point in each (A′, B′, C′), so that all three points are the same distance from O, we shall have a plane triangle, A′ B′ C′, which in some respects corresponds with the spherical triangle, and in some respects differs from it.

The three angles A′, B′, and C′, are the same as the three angles A, B, and C; so that the two triangles are equiangular.

The plane of the triangle A′ B′ C′ is parallel with the plane passing through the three angular points A, B, C, of the spherical triangle.

Each of the sides, *a′*, *b′*, *c′*, of the triangle A′ B′ C′ is less than the corresponding side *a*, *b*, *c*, of the triangle, just as the area of the plane triangle A′ B′ C′ is less than the area of the curved surface of the triangle A B C.

Let the spherical triangle A B C be right-angled at C, then also the plane triangle A′ B′ C′ is right-angled at C′.

Then also—

$$\frac{B'\,C'}{A'\,B'} = \sin A' = \sin A,$$
$$= \cos B' = \cos B.$$

$$\frac{A'\,C'}{A'\,B'} = \sin B' = \sin B,$$
$$= \cos A' = \cos A.$$

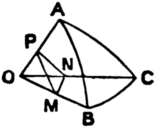
$$\frac{A'\,C'}{B'\,C'} = \tan B' = \tan B,$$
$$= \cot A' = \cot A.$$

$$\frac{B'\,C'}{A'\,C'} = \tan A' = \tan A,$$
$$= \cot B' = \cot B.$$

$$\frac{A'\,B'}{A'\,C'} = \sec A' = \sec A,$$
$$= \operatorname{cosec} B' = \operatorname{cosec} B.$$

$$\frac{A'B'}{B'C'} = \sec B' = \sec B,$$
$$= \operatorname{cosec} A' = \operatorname{cosec} A.$$

So that any one ratio being given, the rest are easily found, both of A and B. C being 90°, its ratios are all known. We have thus a method of finding the ratios for the angles of the triangle. We have now to find those for the sides, which (as we have before shown) are arcs of circles subtending certain angles at the centre, and the ratios of the angles may be also spoken of as ratios of the arcs; *i.e.*, of the sides of the spherical triangle.



To do this, take some point, M in O B, draw M N, in plane B O C, and perpendicular to plane C O A; from N draw N P, in plane C O A, and perpendicular to plane A O B; then join M P, by a line in plane A O B, which will be perpendicular to plane A O C.*

Then we have a triangular pyramid, of which M N P is the base and O the apex, and we have three right-angled triangles, viz.:—

M N O, right-angled at N.
N O P, right-angled at P.
M O P, right-angled at P.

The triangle M N P is quite distinct from the tiangle A′ B′ C′, and is generally in a different plane. But the angles of triangle M N P are the same as those of the spherical triangle.

Now the triangle M O P will give us the ratios of the arc A B; *i.e.*, of the angle A O B:

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* The proof of this is too long to insert here.

Also the triangle M O N will give the ratios of arc B C; *i.e.*, of angle C O B:

And the triangle N O P will give the ratios of arc C A; *i.e.*, of angle C O A.

Thus in plane B O C, the triangle M N O is right-angled at N, and then—

$$\left.\begin{array}{l} \frac{MN}{OM} = \mathit{sin} \\[4pt] \frac{ON}{OM} = \mathit{cos} \\[4pt] \frac{MN}{ON} = \mathit{tan} \\[4pt] \frac{OM}{MN} = \mathit{cosec} \\[4pt] \frac{OM}{ON} = \mathit{sec} \\[4pt] \frac{ON}{MN} = \mathit{cot} \end{array}\right\} \begin{array}{c}\text{of angle B O C, or of arc B C,} \\ \text{or of side } a.\end{array}$$

In same way, in plane A O C, the triangle N O P being right-angled at P, we have—

$$\left.\begin{array}{l} \frac{NP}{ON} = \mathit{sin} \\[4pt] \frac{ON}{NP} = \mathit{cosec} \\[4pt] \frac{OP}{ON} = \mathit{cos} \\[4pt] \frac{ON}{OP} = \mathit{sec} \\[4pt] \frac{NP}{OP} = \mathit{tan} \\[4pt] \frac{OP}{NP} = \mathit{cot} \end{array}\right\} \begin{array}{c}\text{of angle A O C, or of arc C A,} \\ \text{or of side } b.\end{array}$$

Lastly, in plane A O B, the triangle M O P being right-angled at P, we have—

$$\left.\begin{array}{l}\dfrac{MP}{OM} = \sin \\[4pt] \dfrac{OM}{MP} = \operatorname{cosec} \\[4pt] \dfrac{OP}{OM} = \cos \\[4pt] \dfrac{OM}{OP} = \sec \\[4pt] \dfrac{MP}{OP} = \tan \\[4pt] \dfrac{OP}{MP} = \cot \end{array}\right\}\begin{array}{c}\text{of angle } AOB\text{, or of arc } AB\text{,}\\ \text{or of side } c.\end{array}$$

We have thus methods of finding the ratios of all the six angles which have to be taken into account in every solid angle; or, which is the same, in calculating the sides and angles of a spherical triangle.

In the next chapter we shall establish some of the fundamental theorems by means of which we are enabled to solve spherical triangles; *i. e.*, find some of the sides and angles when others are given.

CHAPTER IV.

Fundamental Theorems of Spherical Triangles.

We have seen that every spherical triangle has three angles and three sides, generally expressed by A, B; C, *a*, *b*, *c*. So far spherical and plane triangles are alike.

But while plane triangles have sines, cosines, &c., only for the angles, spherical triangles have these ratios for both angles and sides; since the sides, being arcs of circles, represent angles at the centres of those circles.

The ratios of the six parts of a right-angled spherical triangle (three sides and three angles) are connected by inter-relations expressed by the following six formulæ.

Let A, B, C, be the three angles, C being a right angle.

Let a, b, c, be the sides opposite A, B, C, respectively. Then—

(1.) $\sin a = \sin c \sin A$;

also, $\sin b = \sin c \sin B$.

(2.) $\cos A = \cos a \sin B$;

also, $\cos B = \cos b \sin A$.

(3.) $\cos c = \cos a \cos b$.

(4.) $\cos c = \cot A \cot B$.

(5.) $\tan a = \tan c \cos B$;

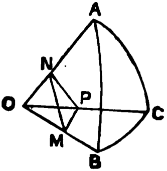
also, $\tan b = \tan c \cos A$.

(6.) $\tan a = \tan A \sin b$;

also, $\tan b = \tan B \sin a$.

Here the six formulæ are stated together, but it is necessary to prove them before we can make use of them. Formulæ 1, 3, 5, and 6, are easily established, and 2 and 4 are derived from them.

Thus, to prove (1)—



$$\sin a = \frac{MP}{OM},$$

$$= \frac{MP}{MN} \cdot \frac{MN}{OM},$$
$$= \sin A \sin c,$$
$$= \sin c \sin A;$$
$$\therefore \sin a = \sin c \sin A.$$

(3.) $$\cos c = \cos a \cos b.$$

For $$\cos c = \frac{ON}{OM},$$
$$= \frac{ON}{OP} \cdot \frac{OP}{OM},$$
$$= \cos b \cos a;$$
$$\therefore \cos c = \cos b \cos a.$$

(5.) $$^{*}\tan b = \tan c \cos A.$$

For, $$\tan b = \frac{PN}{ON},$$
$$= \frac{PN}{MN} \cdot \frac{MN}{NO},$$
$$= \cos A \tan c;$$
$$\therefore \tan b = \tan c \cos A.$$

(6.) $$\tan a = \tan A \sin b.$$

For, $$\tan a = \frac{MP}{OP},$$
$$= \frac{MP}{NP} \cdot \frac{NP}{OP},$$
$$= \tan A \cdot \sin b,$$
$$\therefore \tan a = \tan A \sin b.$$

The second cases may be proved in exactly the same manner, changing, when necessary, the figure, so that instead of M P perpendicular to O P, we have

---

[*] We take the second case, because with our figure the proof is more simple.

P M perpendicular to O M; and instead of M N perpendicular to O N, we have N M perpendicular to O M; also N P perpendicular to O P instead of to O N.

We have now to deduce formulæ (2) and (4) from the theorems already established. Thus, in (4)—

$$\cos c = \cot A \cot B.$$

For, $\tan a = \tan A \sin b,$ (6)

and, $\tan b = \tan B \sin a,$

multiplying these together, we have—

$$\tan a \tan b = \tan A \tan B \sin a \sin b;$$

$$\therefore \frac{\tan a \tan b}{\tan A \tan B} = \sin a \sin b;$$

$$\therefore \frac{1}{\tan A \tan B} = \frac{\sin a \sin b}{\tan a \tan b}.$$

But, $\dfrac{\sin a}{\tan a} = \cos a,$

and, $\dfrac{\sin b}{\tan b} = \cos b,$ $\quad \because \dfrac{\sin}{\cos} = \tan$ (page 39),

$$\therefore \frac{1}{\tan A \tan B} = \cos a \cos b,$$

$$= \cos c \qquad (3)$$

$$\therefore \cot A \cot B = \cos c;$$

$$\therefore \cos c = \cot A \cot B. \qquad (4)$$

To prove (2), we have,—

$$\cos B = \cos a \sin A.$$

For, $\sin a = \sin c \sin A,$ (1)

and, $\tan a = \tan c \cos B;$ (5)

multiplying these crosswise, we have—

$$\sin a \tan c \cos B = \tan a \sin c \sin A;$$

$$\therefore \cos B = \frac{\tan a \sin c \sin A}{\sin a \tan c},$$

$$= \frac{\tan a}{\sin a} \cdot \frac{\sin c}{\tan c} \cdot \sin A,$$

$$= \frac{1}{\cos a} \cdot \cos c \cdot \sin A,$$

$$= \frac{1}{\cos a} \cdot \cos a \cos b \cdot \sin A,$$

$$= \frac{\cos a}{\cos a} \cdot \cos b \cdot \sin A;$$

$$\therefore \cos B = \cos b \sin A;$$

and the other case of the formula

$$\cos A = \cos a \sin A$$

is proved in the same way.

From these fundamental theorems we can obtain many others, which we do not give here, because it is desirable to keep the subject, at this elementary stage, as clear and simple as possible.

In the next chapter we shall discuss the methods of solution of spherical triangles.

CHAPTER V.

SOLUTION OF RIGHT-ANGLED TRIANGLES.

In plane triangles (page 21), if we have three out of the six parts given we can find the other three. If the triangle be right-angled, then two of the five remaining parts will suffice, and we may use the theorem of Euclid, I. 47.

So, in a spherical triangle, if we have three parts given us we can, generally speaking, find the other three, but we cannot, even in right-angled triangles, use Euclid, I. 47, because of the practical difficulties.

But there is this especial peculiarity in spherical triangles, that we have, in reality, *six angles* to find, and then the arcs corresponding to three of these six.

Example.—Given side b and angle A.

Then, of the six terms A, B, C, a, b, c, three, A, b, C, are known, and the substitution of their values in the fundamental equations will generally give the values of the others, or equations from which they may be found.

Thus, to find a,

$$\tan a = \tan A \sin b:$$

to find c,

$$\cos c = \cos a \cos b:$$

to find B,

$$\cos B = \cos b \sin A.$$

Example.—Given B, C, and c, to find A, b, c.

To find a,

$$\sin a = \sin c \sin A:$$

to find c,

$$\cos c = \cos a \cos b:$$

to find A,

$$\cos A = \cos a \sin B.$$

Notice, that to find a, in one example we use

$$\tan a = \tan A \sin b,$$

and in the other,

$$\sin a = \sin c \sin A.$$

This is because in one A and b are known, and in the other A and c. The process of solution is very simple. It is only to select the particular equations which contain on one side the unknown, and on the other only the known, quantities.

But in finding a, b, c, by the sines, tangents, &c., we find, not the sides, but the angles belonging to them: we have then to find the arcs corresponding to these angles, which are the sides of the triangle.

Example.—Let $A = 30°$, $b = 60°$, $C = 90°$.
Required B, a, and c.

To find B,

$$\begin{aligned} \cos B &= \cos b \sin A, \\ &= \cos 60^\circ \sin 30^\circ, \\ &= \tfrac{1}{2} \cdot \tfrac{1}{2}, \\ &= \tfrac{1}{4}, \\ &= \cos 75^\circ\, 31'; \\ \therefore B &= 75^\circ\, 31'. \end{aligned}$$

(Notice that the sum of the three angles is not necessarily 180°.)

To find a,

$$\begin{aligned} \tan a &= \tan A \sin b, \\ &= \tan 30^\circ \sin 60^\circ, \\ &= \frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{2}, \\ &= \tfrac{1}{2}; \end{aligned}$$

$\therefore$ the angle a is the angle whose *tan* is $\tfrac{1}{2}$;

$$\therefore a = 26^\circ\, 33'.$$

To find c,

$$\begin{aligned} \cos c &= \cos a \cos b, \\ &= \cdot 8945446 \times \cdot 5, \\ &= \cos 63^\circ\, 26', \\ \therefore c &= 63^\circ\, 26'. \end{aligned}$$

CHAPTER VI.

OBLIQUE-ANGLED SPHERICAL TRIANGLES.

THE fundamental theorems of oblique-angled spherical triangles are more complex than those of right-angled triangles, just as is the case with plane triangles. To speak more philosophically, however, we should say that the formulæ used for solving spherical triangles generally are numerous and complex, but that in the case of spherical triangles having one right angle we have also other and more simple formulæ. The formulæ for oblique-angled triangles are also true for right-angled triangles, and those in use for right-angled triangles are, in fact, more or less the same, in their simpler forms.

The following is a list of the chief formulæ, given to show its completeness. Each will be proved afterwards.

As before, the three angles are expressed by A, B, C, and the three sides by *a*, *b*, *c*.

Formulæ.

(1.) $$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c};$$

or, $\sin A : \sin B : \sin C :: \sin a : \sin b : \sin c$.

(2.) $$\cos C = \frac{\cos c - \cos a \cos b}{\sin a \sin b}.$$

(3.) $$\sin \frac{C}{2} = \sqrt{\frac{\sin (s-a) \sin (s-b)}{\sin a \sin b}}.$$

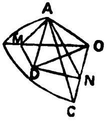
(4.) $$\cos \frac{C}{2} = \sqrt{\frac{\sin s \sin (s-c)}{\sin a \sin b}}.$$

(5.) $$\tan \frac{C}{2} = \sqrt{\frac{\sin (s-a) \sin (s-b)}{\sin s \sin (s-c)}}.$$

(6.) $$\tan \frac{A+B}{2} = \cot \frac{C}{2} \cdot \frac{\cos \frac{1}{2} (a-b)}{\cos \frac{1}{2} (a+b)}.$$

(7.) $$\tan \frac{A-B}{2} = \cot \frac{C}{2} \cdot \frac{\sin \frac{1}{2} (a-b)}{\sin \frac{1}{2} (a+b)}.$$

(8.) $$\cot a \sin c = \cos c \cos B + \sin B \cot A.$$



Let A B C be any spherical triangle.

From A draw A D perpendicular to the plane of B O C.

From D draw D M perpendicular to B O, and D N perpendicular to C O.

Join A M, A N, and D O.

Then also A M O and A N O are right angles.

For A D M and A D O being each 90°;

$$\begin{aligned}\therefore AM^2 &= AD^2 + DM^2,\\ &= (AO^2 - DO^2) + DM^2,\\ &= AO^2 + (DM^2 - DO^2),\\ &= AO^2 + OM^2;\\ \therefore AM^2 &= AO^2 + OM^2,\end{aligned}$$

and A M O is 90°.

Also A N O is 90°.

Then $\dfrac{AM}{AO} = \sin AOB$ or c,

$$\frac{AN}{AO} = \sin AOC \text{ or } b,$$

$$\frac{AD}{AM} = \sin B, \text{ for } AMD = B.$$

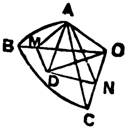
$$\frac{AD}{AN} = \sin C, \text{ for } AND = C.$$

The other ratios of B, C, *b*, *c*, are expressed by means of the same triangles; but those of A and *a* require another, but similar, figure.

But as we have to do, not so much with the actual ratios as with their relations, two angles and two sides will be enough.

We have now to prove the formulæ given above.

(1.) $$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}.$$



$$\frac{AD}{AM} = \sin B;$$

$$\therefore\ AD = AM \sin B;$$

$$\text{also } \frac{AD}{AN} = \sin C;$$

$$\therefore\ AD = AN \sin C;$$

$$\therefore\ AM \sin B = AN \sin C.$$

$$\text{But } \frac{AM}{AO} = \sin c, \text{ and } \frac{AN}{AO} = \sin b;$$

$$\therefore\ AM = AO \sin c, \text{ and } AN = AO \sin b.$$

$$\therefore\ AO \sin c \sin B = AO \sin b \sin c;$$

$$\therefore\ \sin c \sin B = \sin b \sin C;$$

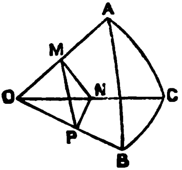
$$\therefore\ \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}.$$

$$\text{In same way, } \frac{\sin A}{\sin a} = \frac{\sin C}{\sin c};$$

$$\therefore\ \frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c};$$

$$\text{or, } \sin A : \sin B : \sin C :: \sin a : \sin b : \sin c.$$

(2.) To find the angles when the three sides are given.



$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}.$$

From M in AO draw MN and MP perpendicular to AO.

Join NP. Then NMO and PMO are right angles.

Then $PN^2 = PM^2 + NM^2 - 2PM.NM \cos PMN$;
also, $PN^2 = PO^2 + NO^2 - 2PO.NO \cos PON$;
subtracting,

$$\therefore \left.\begin{array}{l} PO^2 - PM^2 + NO^2 - NM^2 \\ + 2PM.NM \cos PMN - 2PO.NO \cos PON \end{array}\right\} = 0.$$

But $PMN = A$, and $PON = a$;
also, $PO^2 - PM^2 = OM^2$, and $NO^2 - NM^2 = OM^2$;

$$\therefore 2OM^2 + 2PM.NM \cos A - 2PO.NO \cos a = 0;$$

$$\therefore PN.NM \cos A = PO.NO.\cos a - OM^2;$$

$$\therefore \cos A = \frac{PO.NO}{PN.NM} \cos a - \frac{OM^2}{PN.NM};$$

$$\therefore \cos a = \frac{OM^2}{PO.ON} + \frac{PM.NM}{PO.ON} \cos A,$$

$$= \frac{OM}{OP} \cdot \frac{OM}{ON} + \frac{PM}{OP} \cdot \frac{NM}{ON} \cdot \cos A,$$

$$= \cos b \cos c + \sin b \sin c \cos A;$$

$$\therefore \cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c};$$

which is an expression for the cosine of an angle in terms of the sides.

$$(3.) \qquad \sin \frac{c}{2} = \sqrt{\frac{\sin(s-a) \sin(s-b)}{\sin a \sin b}}.$$

Let $\frac{a+b+c}{2}$ be expressed by s; *i.e.*, let s be the symbol for half the sum of the sides.

Then $1 - \cos A = 1 - \frac{\cos a - \cos b \cos c}{\sin b \sin c}$;

$$\therefore 2 \sin^2 \frac{A}{2} = \frac{\cos b . \cos c + \sin b \sin c - \cos a}{\sin b \sin c}.$$

Because $1 - \cos A = 2 \sin^2 \frac{A}{2}$ (page 97).

Also $\cos (b - c) = \cos b \cos c + \sin b \sin c$ (page 71).

$$\therefore 2 \sin^2 \frac{A}{2} = \frac{\cos (b-c) - \cos a}{\sin b \sin c},$$

$$= \frac{2 \sin \frac{c + a - b}{2} . \sin \frac{a + b - c}{2}}{\sin b \sin c};$$

$$\therefore \sin^2 \frac{A}{2} = \frac{\sin (s-b) . \sin (s-c)}{\sin b . \sin c};$$

$$\therefore \sin \frac{A}{2} = \sqrt{\frac{\sin (s-b) \sin (s-c)}{\sin b \sin c}};$$

which is the same as above, putting A for C.

(4.) $\cos \frac{C}{2} = \sqrt{\frac{\sin s \sin (s-c)}{\sin a \sin b}}$.

As above $1 + \cos C = 2 \cos^2 \frac{C}{2}$,

and $2 \cos^2 \frac{C}{2} = \frac{\cos c (\cos a . \cos b - \sin a \sin b)}{\sin a \sin b}$,

$$= \frac{\cos c - \cos (a+b)}{\sin a . \sin b},$$

$$= \frac{2 \sin \frac{a+b+c}{2} . \sin \frac{a+b-c}{2}}{\sin a . \sin b};$$

$$\therefore \cos^2 \frac{C}{2} = \frac{\sin s . \sin (s-c)}{\sin a . \sin b};$$

$$\therefore \cos\frac{C}{2} = \sqrt{\frac{\sin s \,.\, \sin(s-c)}{\sin a \sin b}}.$$

(5.) $$\tan\frac{C}{2} = \sqrt{\frac{\sin(s-a)\sin(s-b)}{\sin s \sin(s-c)}},$$

$$\tan\frac{c}{2} = \frac{\sin\frac{C}{2}}{\cos\frac{C}{2}};$$

$$= \left\{\frac{\sqrt{\frac{\sin(s-a)\sin(s-b)}{\sin a \sin b}}}{\sqrt{\frac{\sin s \sin(s-c)}{\sin a \sin b}}}\right\}$$

$$= \sqrt{\frac{\sin(s-a)\sin(s-b)}{\sin s \sin(s-c)}}$$

(6.) $$\tan\frac{A+B}{2} = \cot\frac{C}{2}\,.\,\frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)}.$$

For, $$\sin\frac{A+B}{2} = \cos\frac{C}{2}\,.\,\frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}c}.$$

and $$\cos\frac{A+B}{2} = \sin\frac{C}{2}\,.\,\frac{\cos\frac{1}{2}(a+b)}{\cos\frac{1}{2}c};$$

$$\therefore \quad \tan\frac{A+B}{2} = \frac{\cos\frac{C}{2}}{\sin\frac{C}{2}}\,.\,\frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}c},$$

$$= \cot\frac{C}{2}\,.\,\frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}c}.$$

(7.) $\tan\frac{A-B}{2} = \cot\frac{C}{2}\cdot\frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}(a+b)}.$

For, $\sin\frac{A-B}{2} = \cos\frac{1}{2}C \,.\, \frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}c},$

and $\cos\frac{A-B}{2} = \sin\frac{1}{2}C \,.\, \frac{\sin\frac{1}{2}(a+b)}{\sin\frac{1}{2}c};$

$$\therefore \tan\frac{A-B}{2} = \frac{\cos\frac{1}{2}C}{\sin\frac{1}{2}C}\cdot\frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}(a\times b)};$$

$$= \cot\frac{C}{2}\cdot\frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}(a+b)}.$$

(8.) $\cot a \sin c = \cos c \cos B + \sin B \cot A.$

For, $\cos a = \cos b \cos c + \sin b \sin c \cos A;$

also, $\cos b = \cos c \cos a + \sin c \sin a \cos B;$

substituting the value of *cos b* for *cos b*, in the first equation,

$$\cos a = (\cos a \,.\, \cos c + \sin a \sin c \cos B) \cos c + \sin b \sin c \cos A.$$

CHAPTER VII.

SOLUTION OF SPHERICAL TRIANGLES.

BESIDES the formulæ established in the last chapter, we have also some fundamental theorems which are of use in the solution of triangles.

These theorems will also illustrate the difference between spherical and plane triangles in many important points, as well as show that in some they are alike.

Theorems.

Any two sides of a spherical triangle are together greater than the third.

The greater side of a spherical triangle is opposite the greater angle.

Any two angles of a spherical triangle are together less than the third angle + 180°.

In these respects spherical and plane triangles are alike: in the following we find them differ.

The sum of the three angles of a spherical triangle is greater than two and less than six right angles.

The sum of the three sides is less than 360°.

The sum of two sides is 180° *if the sum of the opposite angles be* 180°, *greater if it be greater, and less if it be less;* i.e., *the sum of any two sides are greater than, equal to, or less than,* 180°, *according as the sum of the opposite angles are greater than, equal to, or less than,* 180°.

The sum of the three sides are less than 360°; i.e., *the three angles of the solid angle at the centre of the sphere are together less than* 360°, *or four right angles.*

By means of the formulæ of last chapter and the above theorems, we are able to find all the sides and angles of spherical triangles if three out of the six be given us. In right-angled triangles (*i.e.*, triangles having one angle = 90°), we require only two parts in addition to the right angle.

Solution of Right-angled Triangles.

Case 1. If one side and one angle be given.

Let A and a be given ($c=90°$) to find B, b, and c.

Then,
$$\sin B = \frac{\cos A}{\cos a}:$$
which gives B.

$$\sin b = \frac{\tan a}{\tan A}:$$
which gives b.

$$sin\ c = \frac{sin\ a}{sin\ A}:$$

which gives c.

Thus we find B, b, and c, from A and a.

Example.—Let $A = 25°$, $a = 13°$, $C = 90°$.

Required B, b, and c.

Notice, that all the parts (angles and sides) are expressed in degrees. If the actual length of the side be required, it can be found, by calculating what portion of the circumference it is.

Then,
$$sin\ B = \frac{cos\ A}{cos\ a};$$
$$\therefore sin\ B = \frac{cos\ 25°}{cos\ 13°};$$
$$\begin{aligned} \therefore L\ sin\ B &= L\ cos\ 25° - L\ cos\ 13° + 10, \\ &= 9 \cdot 9572757 - 9 \cdot 9887239 + 10, \\ &= 9 \cdot 9685518, \\ &= L\ sin\ 68°\ 27' + ; \end{aligned}$$
$$\therefore B = 68°\ 27' + ;$$

i.e., B is between 68° 27′ and 68° 28′.

Notice, that 10 has to be added to one side of the equation, because $L\ sin\ B = l\ sin\ B + 10$; while the two 10's on the other side balance each other. Thus—

$$L\ sin\ B = L\ cos\ 25° - L\ cos\ 13°, \text{ is the same as}$$
$$\log sin\ B + 10 = \log cos\ 25° + 10 - \log cos\ 13° - 10.$$

$$sin\ b = \frac{tan\ a}{tan\ A};$$
$$\begin{aligned} \therefore L\ sin\ b &= L\ tan\ a - L\ tan\ A + 10, \\ &= L\ tan\ 13° - L\ tan\ 25° + 10, \\ &= L\ sin\ 26°\ 20' + ; \end{aligned}$$

$$\therefore b = 26^\circ\ 20' +,$$

$$\sin c = \frac{\sin a}{\sin A};$$

$$\begin{aligned}\therefore L \sin c &= L \sin a - L \sin A + 10,\\ &= L \sin 13^\circ - L \sin 25^\circ + 10,\\ &= L \sin 32^\circ\ 9' +;\\ \therefore c &= 32^\circ\ 9' +.\end{aligned}$$

Therefore,

$$\begin{aligned}A &= 25^\circ,\\ B &= 68^\circ\ 27',\\ C &= 90^\circ.\\ a &= 13^\circ,\\ b &= 26^\circ\ 20',\\ c &= 32^\circ\ 9';\end{aligned}$$

which is the complete statement of the triangle.

Example.—Let $B = 17^\circ$ and $a = 41^\circ$.
Required, A, b, and c.
Then,

$$\begin{aligned}\cos A &= \cos a \sin B,\\ \tan b &= \tan B \sin a,\\ \cos c &= \cos a \cos b.\end{aligned}$$

Thus,

$$\begin{aligned}\cos A &= \cos a \sin B,\\ &= \cos 41^\circ \sin 17^\circ;\\ \therefore L \cos A &= L \cos 41^\circ + L \sin 17^\circ - 10,\\ &= 9{\cdot}8777799 + 9{\cdot}4659353 - 10,\\ &= L \cos 77^\circ\ 16' +;\\ \therefore A &= 77^\circ\ 16'.\end{aligned}$$

$$\begin{aligned}\tan b &= \tan B \sin a,\\ &= \tan 17^\circ \sin 41^\circ;\end{aligned}$$

$$\begin{aligned}
\therefore \text{L}\,tan\,b &= \text{L}\,tan\,17^\circ + \text{L}\,sin\,41^\circ - 10,\\
&= 9{\cdot}4853390 + 9{\cdot}8169429 - 10,\\
&= \text{L}\,tan\,11^\circ\,20' + ;\\
\therefore b &= 11^\circ\,20'.
\end{aligned}$$

$$\begin{aligned}
cos\,c &= cos\,a\,cos\,b,\\
&= cos\,41^\circ\,cos\,11^\circ\,20',\\
\text{L}\,cos\,c &= \text{L}\,cos\,41^\circ + \text{L}\,cos\,11^\circ\,20' - 10,\\
&= 9{\cdot}8777799 + 9{\cdot}9914478 - 10,\\
&= \text{L}\,cos\,42^\circ\,15'.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\text{A} &= 77^\circ\,16',\\
\text{B} &= 17^\circ,\\
\text{C} &= 90^\circ,\\
a &= 41^\circ,\\
b &= 11^\circ\,20',\\
c &= 42^\circ\,15',
\end{aligned}$$

and the triangle is completely known.

In the same way all the other cases of right-angled spherical triangles may be solved, by the use of the formulæ in Table 8.

Solution of Oblique Spherical Triangles.

The formulæ used in the solution of oblique spherical triangles are given and proved on p. 139, but are repeated here for convenience of reference.

(1.) $$\frac{sin\,\text{A}}{sin\,a} = \frac{sin\,\text{B}}{sin\,b} = \frac{sin\,\text{C}}{sin\,c}.$$

(2.) $$cos\,c = cos\,a\,cos\,b + sin\,a\,sin\,b\,cos\,\text{C}.$$

(3.) $$cot\,a\,sin\,c = cos\,c\,cos\,\text{B} + sin\,\text{B}\,cot\,\text{A}.$$

(4.) $$tan\,\frac{\text{A}+\text{B}}{2} = cot\,\frac{\text{C}}{2}\cdot\frac{cos\,\frac{1}{2}\,(a-b)}{cos\,\frac{1}{2}\,(a+b)}.$$

$$(5.)\quad \tan\frac{A-B}{2} = \cot\frac{C}{2}\cdot\frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}(a+b)}.$$

$$(6.)\quad \sin\frac{C}{2} = \sqrt{\frac{\sin(s-a)\sin(s-b)}{\sin a\sin b}}.$$

$$(7.)\quad \cos\frac{C}{2} = \sqrt{\frac{\sin s\sin(s-c)}{\sin a\sin b}}.$$

$$(8.)\quad \tan\frac{C}{2} = \sqrt{\frac{\sin(s-a)\sin(s-b)}{\sin s\,.\,\sin(s-c)}},$$

$$= \frac{M}{\sin(s-c)}.$$

In the above, $s = \dfrac{a+b+c}{2}$,

$$M = \frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}.$$

Case 1.—If a, b, and C, be given to find A, B, and c.

Then, $$\tan\frac{A}{2} = \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s\sin(s-a)}};$$

and $$\tan\frac{B}{2} = \sqrt{\frac{\sin(s-a)\sin(s-c)}{\sin s\sin(s-b)}}:$$

which would give A and B, but that c is unknown.

But $$\tan\frac{A+B}{2} = \cot\frac{C}{2}\cdot\frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)};$$

and $$\tan\frac{A-B}{2} = \cot\frac{C}{2}\cdot\frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}(a+b)};$$

which expressions give the sum of A and B, and also their difference, in terms of C, a, and b, all of which are known. A + B and A − B being known, A and B are easily found.

The third side, c, is found thus :—

$$\frac{\sin c}{\sin b} = \frac{\sin C}{\sin B};$$

$$\therefore \sin c = \frac{\sin C}{\sin B} \cdot \sin b;$$

which gives c, and the triangle is known entirely.

Case 2.—If A, B, and c, be given to find C, a, and b.

Then, $$\tan\frac{a+b}{2} = \tan\frac{c}{2} \cdot \frac{\cos\frac{A-B}{2}}{\cos\frac{A+B}{2}},$$

and $$\tan\frac{a-b}{2} = \tan\frac{c}{2} \cdot \frac{\sin\frac{A-B}{2}}{\sin\frac{A+B}{2}};$$

which gives $a+b$ and $a-b$ in terms of A, B, and c, all of which are known, and therefore gives a and b.

To find C, we have,—

$$\frac{\sin C}{\sin B} = \frac{\sin c}{\sin b};$$

$$\therefore \sin C = \frac{\sin c}{\sin b} \cdot \sin B.$$

which gives C, and the triangle is entirely known.

TABLE 1.—RELATIONS BETWEEN RATIOS.

	Sine.	*Cosecant.*	*Cosine.*	*Secant.*	*Tangent.*	*Cotangent.*
Sine . .	1	$\frac{1}{cosec}$	$\sqrt{1-cos^2}$	$\frac{\sqrt{sec^2-1}}{sec}$	$\frac{tan}{\sqrt{tan^2+1}}$	$\frac{1}{\sqrt{1+cot^2}}$
Cosecant .	$\frac{1}{sin}$	1	$\frac{1}{\sqrt{1-cos^2}}$	$\frac{sec}{\sqrt{sec^2-1}}$	$\frac{\sqrt{tan^2+1}}{tan}$	$\sqrt{1+cot^2}$
Cosine .	$\sqrt{1-sin^2}$	$\frac{\sqrt{cosec^2-1}}{cosec}$	1	$\frac{1}{sec}$	$\frac{1}{\sqrt{1+tan^2}}$	$\frac{cot}{\sqrt{1+cot^2}}$
Secant .	$\frac{1}{\sqrt{1-sin^2}}$	$\frac{cosec}{\sqrt{cosec^2-1}}$	$\frac{1}{cos}$	1	$\sqrt{1+tan^2}$	$\frac{\sqrt{1+cot^2}}{cot}$
Tangent .	$\frac{sin}{\sqrt{1-sin^2}}$	$\frac{1}{\sqrt{cosec^2-1}}$	$\frac{\sqrt{1-cos^2}}{cos}$	$\sqrt{sec^2-1}$	1	$\frac{1}{cot}$
Cotangent	$\frac{\sqrt{1-sin^2}}{sin}$	$\sqrt{cosec^2-1}$	$\frac{cos}{\sqrt{1-cos^2}}$	$\frac{1}{\sqrt{sec^2-1}}$	$\frac{1}{tan}$	1

TABLE 2.

FACTS AND FORMULÆ.

		Reference Page.
$Sin = \frac{1}{cosec}$. $cos = \frac{1}{sec}$. $tan = \frac{1}{cot}$		43
$Sec = \frac{1}{cos}$. $cosec = \frac{1}{sin}$. $cot = \frac{1}{tan}$		43
Sin *Cos*	always between 1 and -1; *i. e.*, never *greater* than 1	58
Sec *Cosec*	always between 1 and ∞; *i. e.*, never *less* than 1	58
Tan *Cotan*	may be any magnitude; *i. e.*, varies from $0 - \infty$	58

TABLE 3.

Magnitudes of	0°	90°	180°	270°	360°
Sine	0	1	0	−1	0
Cosine . . .	1	0	−1	∞	1
Tangent . . .	0	∞	0	∞	0
Secant	1	∞	−1	∞	1
Cosecant . . .	∞	1	∞	−1	∞
Cotangent . .	∞	0	∞	0	∞

TABLE 4.

Signs of the Ratios.	*Sine.*	*Cosecant.*	*Secant.*	*Cosine.*	*Tangent.*	*Cotangent.*
Between 0° & 90°	+	+	+	+	+	+
Between 90° & 180°	+	+	−	−	−	−
Between 180° & 270°	−	−	−	−	+	+
Between 270° & 360°	−	−	+	+	−	−

TABLE 5.

—	15°	30°	45°	60°	90°	180°	270°	360°
Sin . . .	$\frac{\sqrt{3}-1}{\sqrt{8}}$	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	−1	0
Cos . . .	$\frac{1+\sqrt{3}}{\sqrt{8}}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	−1	0	1
Tan . . .	$\frac{\sqrt{3}-1}{1+\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞	0	∞	0
Cosec . . .	$\frac{\sqrt{8}}{\sqrt{3}-1}$	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1	∞	−1	∞
Sec . . .	$\frac{\sqrt{8}}{1+\sqrt{3}}$	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	∞	−1	∞	1
Cot . . .	$\frac{1+\sqrt{3}}{\sqrt{3}-1}$	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	3	∞	0	∞

Table 6.

Ratios of (A + B) expressed in terms of A and B:—

		Reference Page.
Sin	$= \sin A . \cos B + \cos A . \sin B$.	69
Cos	$= \cos A . \cos B - \sin A . \sin B$.	69
Tan	$= \dfrac{\tan A + \tan B}{1 - \tan A . \tan B}$	
Cot	$= \dfrac{1 - \tan A . \tan B}{\tan A + \tan B}$	
Sec	$= \dfrac{1}{\cos A . \cos B - \sin A . \sin B}$.	
Cosec	$= \dfrac{1}{\sin A . \cos B + \cos A . \sin B}$.	

Table 7.

Ratios of (A − B) expressed in terms of A and B:—

		Reference Page.
Sin	$= \sin A . \cos B - \cos A . \sin B$.	71
Cos	$= \cos A . \cos B + \sin A . \sin B$.	71
Tan	$= \dfrac{\tan A - \tan B}{1 + \tan A . \tan B}$	
Cot	$= \dfrac{1 + \tan A . \tan B}{\tan A - \tan B}$	
Sec	$= \dfrac{1}{\cos A . \cos B + \sin A . \sin B}$.	
Cosec	$= \dfrac{1}{\sin A . \cos B - \cos A . \sin B}$.	

Table 8.

Facts and Formulæ for the Solution of Triangles.

	Reference Page.
$\frac{\text{Circumference}}{\text{Diameter}} = 3\frac{1}{7} = \frac{355}{113} = 3{\cdot}14159 = \pi$.	75
$\frac{\text{Circumference}}{\text{Radius}} = \frac{360^\circ}{\text{radius}} = 6\frac{2}{7} = 6{\cdot}28318$.	75
And $\frac{360^\circ}{6{\cdot}28318} = 57{\cdot}29577 = \omega^\circ$.	75
π expresses the ratio of circumference to diameter	76
ω° expresses the number of degrees contained in an angle subtended by an arc equal in length to the radius	76
$\sin A : \sin B : \sin C :: a : b : c$	60
$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$	61
$s = \frac{1}{2}(a + b + c)$.	
$\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}} . \sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$	

Area of triangle—

$$= \frac{1}{2} bc \sin A = \sqrt{s(s-a)(s-b)(-c)}$$

TABLE 9.

EQUIVALENT VALUES.

	Reference Page.
$\mathit{Sin}\ A = 2\,\mathit{sin}\,\frac{1}{2}A\,\mathit{cos}\,\frac{1}{2}A$	96
$\mathit{Cos}\ A = \mathit{cos}^2\,\frac{1}{2}A - \mathit{sin}^2\,\frac{1}{2}A$	96
$= 2\,\mathit{cos}^2\,\frac{1}{2}A - 1$	96
$= 1 - 2\,\mathit{sin}^2\,\frac{1}{2}A$	96
$\mathit{Tan}\ A = \dfrac{2\,\mathit{tan}\,\frac{1}{2}A}{1 - \mathit{tan}^2\,\frac{1}{2}A}$	96
$\mathit{Cosec}\ A = \dfrac{1}{2\,\mathit{sin}\,\frac{1}{2}A\,\mathit{cos}\,\frac{1}{2}A}$	96
$\mathit{Sec}\ A = \dfrac{1}{\mathit{cos}^2\,\frac{1}{2}A - \mathit{sin}^2\,\frac{1}{2}A}$	96
$\mathit{Cot}\ A = \dfrac{1 - \mathit{tan}^2\,\frac{1}{2}A}{2\,\mathit{tan}\,\frac{1}{2}A}$	96

$\mathit{Sin}\ 2A = 2\,\mathit{sin}\,A\,\mathit{cos}\,A$	94
$\mathit{Cos}\ 2A = \mathit{cos}^2\,A - \mathit{sin}^2\,A$	94
$= 2\,\mathit{cos}^2\,A - 1$	94
$= 1 - 2\,\mathit{sin}^2\,A$	95
$\mathit{Tan}\ 2A = \dfrac{2\,\mathit{tan}\,A}{1 - \mathit{tan}^2\,A}$	95
$\mathit{Cosec}\ 2A = \dfrac{1}{2\,\mathit{sin}\,A\,\mathit{cos}\,A}$	95
$\mathit{Sec}\ 2A = \dfrac{1}{\mathit{cos}^2\,A - \mathit{sin}^2\,A}$	95
$\mathit{Cot}\ 2A = \dfrac{1 - \mathit{tan}^2\,A}{2\,\mathit{tan}\,A}$	96

	Reference Page.
$Sin\ A = \sqrt{\tfrac{1}{2}(1 - cos\ 2A)}$	97
$Cos\ A = \sqrt{\tfrac{1}{2}(1 + cos\ 2A)}$	97
$Tan\ A = \sqrt{\dfrac{1 - cos\ 2A}{1 + cos\ 2A}}$	97
$Cosec\ A = \dfrac{1}{\sqrt{\tfrac{1}{2}(1 - cos\ 2A)}}$	97
$Sec\ A = \dfrac{1}{\sqrt{\tfrac{1}{2}(1 + cos\ 2A)}}$	97
$Cot\ A = \sqrt{\dfrac{1 + cos\ 2A}{1 - cos\ 2A}}$	97

$Sin\ A = \dfrac{\sqrt{(1 + sin\ 2A)} + \sqrt{(1 - sin\ 2A)}}{2}$	98
$Cos\ A = \dfrac{\sqrt{(1 + sin\ 2A)} - \sqrt{(1 - sin\ 2A)}}{2}$	98
$Tan\ A = \dfrac{\sqrt{(1 + sin\ 2A)} + \sqrt{(1 - sin\ 2A)}}{\sqrt{(1 + sin\ 2A)} - \sqrt{(1 - sin\ 2A)}}$	98
$Cosec\ A = \dfrac{2}{\sqrt{(1 + sin\ 2A)} + \sqrt{(1 - sin\ 2A)}}$	98
$Sec\ A = \dfrac{2}{\sqrt{(1 + sin\ 2A)} - \sqrt{(1 - sin\ 2A)}}$	98
$Cot\ A = \sqrt{\dfrac{1 - cos\ 2A}{1 + cos\ 2A}}$	98

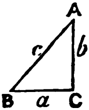
APPENDIX.

PART I.

EXERCISES AND PROBLEMS.

CHAPTER II.

In the right-angled triangle, A B C.



(1.) $b = 7$ and $c = 14$. Required a, A, and B.
(2.) $b = 4$ and $a = 4$. Required c, A, and B.
(3.) $a = 3$ and $c = 6$. Required b, A, and B.
(4.) $b = 1\cdot74$ and $a = 3$. Required b, A, and B.
(5.) $a = 3\cdot48$ and $c = 4$. Required b, A, and B.
(6.) $a = 3\cdot48$ and $c = 400$. Required b, A, and B.

(7.) A ladder 14·1 feet long made an angle of 45° with the side of a house. Required the remaining angle and the sides of the triangle so formed.

(8.) A string 20 feet long, tied to the top of a tree, reached the ground at 10 feet from the root. Required the remaining side and angles of the triangle formed by the tree, string, and ground.

(9.) A boat crossing a stream 100 yards wide, made an angle of 30° with the direct route. Required the triangle formed by the lines of route and opposite bank.

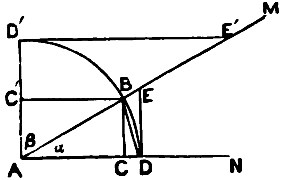
CHAPTER III.

$$\left.\begin{array}{lll}
(1.) & \textit{Sine} & = \frac{1}{3} \\
(2.) & \textit{Cosine} & = \frac{1}{2} \\
(3.) & \textit{Tangent} & = 4 \\
(4.) & \textit{Sine} & = \frac{1}{9} \\
(5.) & \textit{Cosecant} & = 2 \\
(6.) & \textit{Cotangent} & = \frac{1}{4} \\
(7.) & \textit{Secant} & = 2 \\
(8.) & \textit{Sine} & = -\frac{1}{2} \\
(9.) & \textit{Cosine} & = -\frac{1}{4} \\
(10.) & \textit{Tangent} & = -\frac{1}{2} \\
(11.) & \textit{Secant} & = -7 \\
(12.) & \textit{Cosecant} & = -6
\end{array}\right\} \text{Find the values of the remaining ratios.}$$

The negative values are given to accustom the mind to them, and to give facility in working them.

(13.) *Tan* = 7·784, find *sin* and *sec.*
(14.) *Sin* = ·47, find *cot* and *cos.*
(15.) *Tan* = 7·41, find *cos* and *cosec.*
(16.) *Sec* = 4·8, find *sin* and *tan.*
(17.) *Cos* = ·57, find *tan* and *sec.*
(18.) *Cosec* = 4·84, find *tan* and *cos.*
(19.) *Sec* = 3·6, find *sin* and *tan.*
(20.) *Cot.* = 7·4, find all the other ratios.

CHAPTER IV.



(1.) $AB = 4$, $AE = 5$, $BC = 1$. Find AC, AD, and DE.

(2.) $AB = 5$, $CD = 1$. Find AC, BE, BC, and DE.

(3.) $AC = DC$. Find the value of $\frac{AB}{AE}$ and of $\frac{BC}{DE}$.

(4.) What relation exists between DE and BC, and when are they equal?

(5.) Find the magnitude of α and β, when $D'E' = DE$.

(6.) Find the magnitude of angle ABC, when $AE' = 2\,AD'$.

(7.) Prove $\beta = 30°$, when $AB = BD$.

(8.) $AB = 5$. Find AC, AE, and DE, if $\beta = 30°$.

(9.) If $\sin\alpha = \cdot484$, find *cos*, *tan*, and *sec* of β.

(10.) If $\cos\alpha = \cdot574$, find *tan* and *cosec* of β.

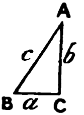
(11.) If $\tan\alpha = 4\cdot79$, find *sin* and *cos* of β.

(12.) If $\cot\alpha = 6\cdot84$, find *sec* and *sin* of β.

(13.) If $\sec\alpha = 7\cdot9$, find *tan* and *cosec* of β.

(14.) If $\operatorname{cosec}\alpha = 2\cdot57$, find all the ratios of β.

CHAPTER V.



(1.) If $a = 3, b = -4$. Find the values of A and B.

(2.) If $a = -3,\ b = -4$. Find the values of A and B.

(3.) $Sin = \frac{1}{2}$, $cos = -\frac{\sqrt{3}}{2}$. Find the magnitude of the angle, and the remaining ratios.

(4.) $Sec = 2$, $tan = -\sqrt{3}$. Find the magnitude of the angle, and the remaining ratios.

(5.) $Cot = -2\ cos$. Find the other ratios and the angle.

(6.) $Sin = \frac{tan}{2}$. Find the other ratios and the angle.

(7.) Find all the ratios of 240°.

(8.) Find *sin* and *cot* of 375°.

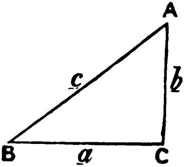
(9.) Find *tan* and *sec* of 285°.

(10.) Find *cot*, *sec*, and *sin* of 165°.

(11.) Find all the ratios of 300°.

(12.) Find all the ratios of 120°.

CHAPTER VI.



In the above triangle—

(1.) $c = 4, b = 2$. Find a, A, and B.

(2.) $a = 17{\cdot}4, c = 20$. Find b, A, and B.

(3.) $b = 2, c = \sqrt{8}$. Find a, A, and B.

(4.) $a = 47$ feet, $b = 39$ feet. Find A, B, and c.

(5.) $a = 2{\cdot}75, c = 3{\cdot}5$ feet. Find A, B, and b.

(6.) $a = 1{\cdot}5$ inches, A = 14°. Find B, b and c.

(7.) $c = 7$ inches, A = 24°. Find B, a and b.

(8.) $a = 1$ mile, $b = 795$ yards, C = 90°. Find A, B and c.

(9.) $a = {\cdot}49$ of a mile, A = 17°. Find B, b and c.

(10.) $a = 9$ inches, $c = 18$ inches. Find A, B and b.

(11.) $c = 47$ yards, A = 4°. Find B, a and b.

CHAPTER VII.

(1.) Find the values for *sine* and *cosine* of 120°, in the terms of the ratios of 90° and 30°.

(2.) Find the above also in terms of 60° and 60, and of 180 and 60°.

(3.) Find the ratios of 90° in terms of 45° and 45°.

(4.) Find the *sine, cosine,* and *secant* of 60° in terms of 90° and 30°.

(5.) Express the *sine, secant,* and *tangent,* of 270° in terms of 180° and 90°.

(6.) Express the *cosine* and *tangent* of 180° in terms of 90° and 90°.

CHAPTER VIII.

(1.) Express 14·741 grades in degrees, minutes, &c.

(2.) Express 14° 14′ 14″ in grades, minutes, &c.

(3.) In a right-angled triangle, the perpendicular equals half the hypothenuse. Express the angles in degrees, and in grades.

(4.) Find the ratio between 67° 14′ and 77^g 14′.

(5.) Express 101 grades in degrees, minutes, &c.

(6.) Express 84° in grades, minutes, &c.

CHAPTER IX.

(1.) Express 36° 10′ as a fraction of ω°.

(2.) In a right-angled triangle, one acute angle is $\frac{3}{4}$ of the other. Find the three angles, and express them in English, French, and circular methods.

(3.) Express $\frac{\omega^\circ}{4}$ in grades, also in degrees.

(4.) A circle has a radius of 8·47. Find the ratio of an arc 4·27 to the circumference.

CHAPTER X.

(1.) The *sine* of an angle = ·3784 of the radius. Find the angle.

(2.) In a right-angled triangle the perpendicular is $\frac{1}{5}$ of the hypothenuse. Find the three angles.

(3.) The base of a right-angled triangle = $\frac{1}{5}$ hypothenuse. Find the three angles.

(4.) $a = 49 \cdot 4$, $A = 90°$, $B = 23° 34'$. Solve the triangle.

CHAPTER XI.

(1.) Given $C = 14° 24'$, $a = 47 \cdot 8$, $b = 39 \cdot 4$. Find A, B, and c.

(2.) Given $B = 17° 34' 18''$, $a = 23 \cdot 94$, $c = 17 \cdot 14$. Find A, C, and b.

(3.) Given $A = 39° 14' 29''$, $b = 87 \cdot 4$, $c = 84 \cdot 3$. Find B, C, and a.

(4.) Given $A = 14° 3' 41''$, $b = 84 \cdot 41$, $a = 32 \cdot 7$. Find B, C, and c.

(5.) Given $B = 37° 8' 32''$, $a = 39 \cdot 47$, $b = 84 \cdot 32$. Find A, C, and c.

(6.) Given $C = 41° 13' 42''$, $b = 14 \cdot 27$, $c = 39 \cdot 41$. Find A, B, and a.

(7.) Given $A = 14° 31' 14''$, $B = 80° 1' 14''$, $c = 39 \cdot 78$. Find C, a, and b.

(8.) Given $A = 89° 14' 14''$, $C = 30° 40'$, $b = 41 \cdot 69$. Find B, a, and c.

PART II.

GENERAL EXAMPLES.

(1.) Prove $\frac{sin}{cos} = \frac{1}{\sqrt{cosec^2 - 1}}$.

(2.) Prove $cot^2 . cos^2 = cot^2 - cos^2$.

(3) Prove $sec^2 \times cosec^2 = sec^2 + cosec^2$.

(4.) Prove $tan\, 60^\circ = \frac{tan\ 20^\circ + tan\ 40^\circ}{1 - tan\ 20^\circ\ tan\ 40^\circ}$.

(5.) Prove $(tan\ 45^\circ + A) = \frac{1 + tan\ A}{1 - tan\ A}$.

(6.) Prove $\frac{1 + sin\ A}{cos\ A} = tan\left(45^\circ + \frac{A}{2}\right)$.

(7.) Given $a = 14 \cdot 713$, $c = 7 \cdot 81$, $B = 47^\circ$.

(8.) Given b, c, and B, to find A, C, and a.

(9.) $A = 49^\circ\ 1'$, $B = 37^\circ\ 5'$, $c = 37 \cdot 4$. Find C, a, and b.

(10.) Given $B = 95^\circ\ 3'$, $C = 14^\circ\ 32'$, and $b = 7 \cdot 8$. Find A, a, and c.

(11.) A castle is surrounded by a moat. Find the height of the castle and the width of the moat.

(12.) A C is a flagstaff on the top of a tower C D. Required the height of each.

(13.) To find the width of a marsh without crossing it.

(14.) The spire of a church, 200 feet high, subtends at A, on the north, an angle of 24°; and at B, on the east, an angle of 31°. Required the distance of A from B.

(15.) From the ratios of 90° and 60°, find those of 150° and 30°.

(16.) From the ratios of 90° and 30°, find those of 120° and 60°.

(17.) From the ratios of 60° and 30°, find those of 90°.

(18.) From the ratios of 90° and 45°, find those of 135°.

(19.) From the ratios of 60° and 45°, find those of 105° and 15°.

(20.) From the ratios of 30° and 45°, find those of 75° and − 15°.

(21.) If the diameter of a circle be 10 feet, what angle does an arc of 7 feet subtend?

(22.) If the radius be 8 feet, what is the length of the arc subtending an angle of 49°?

(23.) If the length of an arc be 8 inches, and the angle 4°, what is the length of the radius?

(24.) In a circle of 12 feet radius, find the number of degrees in an arc of 40 inches, and the length of an arc of 40°.

(25.) The circumference of a circle is 18 feet; find the length of an arc of 80°, and the number of degrees in an arc of 4 feet.

(26.) In each of two circles, one of 3 inches radius, and the other of 4 feet, take an arc of 20°, and compare the lengths.

LONDON: PRINTED BY WILLIAM CLOWES AND SONS, STAMFORD STREET
AND CHARING CROSS.